Hedging of Interest Rate Derivatives

Cash and the Zero Curve

The simplest contract is a unit notional, zero-coupon bond to be paid at time T (the *maturity*). The value of such a bond is denoted by $P(T)$.[a]

The function P thus describes the evolution through time of interest rate expectations.

The instantaneous *forward rate* f is defined by $f(T) \equiv -P'(T)/P(T)$. Thus, the *forward curve* f provides a local view of the market forecast for future interest rates. While knowledge of P and f are in principle equivalent, the latter provides a superior framework for practical analysis.

FRAs, Swaps, and Bond Equivalence

A forward rate agreement, *FRA*, is a contract to lend at a previously agreed rate over some time interval—thus it is equivalent to a calendar spread of zero-coupon bonds. A swap is very similar to a succession of FRAs, so its price can nearly be determined from the zero curve. The slight differences between Libor swaps and coupon-paying bonds stem from the differences between the Libor end date and the period payment date, and also (in most currencies) the difference between fixed and floating payment frequencies. For a more detailed description, *see* **LIBOR Rate**.

Libor Futures

A Libor futures contract (*see* **Eurodollar Futures and Options**) pays, at its settlement, a proportion of the Libor rate fixed on the futures expiry date. However, since an FRA makes its payment only at its maturity date, its par rate in a risk-neutral world is equal to the expectation under the discount-adjusted measure to that maturity date. The daily updating of posted margins for Libor futures means that profit or loss from rate fluctuations is realized immediately; thus the par futures price reflects the risk-neutral expectation in an undiscounted measure.

Because the resulting "futures convexity adjustment" does not closely track other measures of volatility, it is traded actively only by a few specialists. For most purposes, we can think of a future as being equivalent to an FRA plus an exogenously specified spread.

Yield Curve Construction

Since the function $P(T)$, or equivalently $f(T)$, practically determines the value of these Libor-based instruments, we price less-liquid instruments by fitting a *yield curve*—any object from which P and f can be computed—to the observed values of the most liquid *build instruments*. Since there will not be above a few dozen such instruments, this fitting problem is severely underconstrained.

One common method is *bootstrapping* of *zero yields*: we specify that the yield curve will be defined by linear interpolation on the zero-coupon bond yield $y(T) \equiv -\ln P(T)/T$. This restricts the curve's degrees of freedom to one per interpolation point. If we place one interpolation point at the last maturity date (the latest payment or rate end date) of each build instrument, we can solve for each corresponding value of y with a succession of one-dimensional root searches.

Since $f(T) = y(T) + Ty'(T)$, the forward curve thus constructed is gratuitously discontinuous and contains large-scale interpolation artifacts. We do not wish to recommend this construction method or to disparage others, but only wish to note its frequent use and to show a concrete example. For a more complete discussion, *see* **Yield Curve Construction**.

Hedging on the Yield Curve

Once a yield curve is built, it can be used to price similar trades that are not among the build instruments, such as forward-starting or nonstandard swaps. Such pricing depends on two implicit assumptions: that the yield curve is the underlying of these trades as well as of its build instruments, and that its interpolation methods (or other nonmarket constraints) are sufficiently accurate. In practice, the former is widely accepted for Libor-based products, while the latter is a major arena of competition among market makers.

Any trade priced on the yield curve will have a *forward rate risk*, which we denote by $\delta_f(T)$, so that its change in value for a small curve fluctuation Δ_f is equal to the change in $\delta_f^\wedge \Delta f \equiv \int_0^\infty \delta_f(u)\Delta_f(u)\,\mathrm{d}u$. Formally, we write

$$\delta_f(t) \equiv \lim_{h\to 0,\epsilon\to 0} \frac{U(f + h\eta_{t,\epsilon}) - U(f)}{h\epsilon} \quad (1)$$

where

- $U(f)$ is the trade's value for a given yield curve described by the forward rates f;
- $\eta(z)$ is a $\mathcal{C}^\infty$ test function with support in [0, 1] and $\int_0^1 \eta = 1$; and
- $\eta_{t,\epsilon}(z) \equiv \eta((z - t)/\epsilon)$.

For the *linear* trades, that we have so far discussed, δ_f will change very little as f changes. A portfolio of trades with no net δ_f has, at least for that moment, no interest rate risk.

Figure 1 shows the forward rate risk for a swap. The large-scale behavior is unsurprising; the forward rate risk steadily decreases as coupons are paid. The small-scale spikes are caused by overlapping, or in one case underlapping, of the start and end dates for the Libor rates on the floating side. The vertical scale is, of course, proportional to the swap notional amount, and is not shown here.

In practice, especially when trades cannot be exactly represented by equivalent cash flows, we will not know δ_f exactly but we will have rather a numerically computed (e.g., piecewise constant) approximation thereto; but since we can control the *buckets*, that is, the intervals over which δ_f is kept constant, this is not a major difficulty.

Response Functions

However, we cannot execute a hedge of the forward rate risk directly; instead, we must choose a set of *hedge instruments* that will allow us to offset it. Often, these hedge instruments are exactly the build instruments. Each hedge instrument will also, of course, have a forward rate sensitivity. In practice, we generally consider the sensitivity, not of the instrument value, but of the *implied par rate* (or just *implied rate*): The implied FRA rate for futures, par coupon for swaps, or yield for bonds implied by the yield curve.

In this case, we can compute a hedge by slightly bumping each instrument's implied rate r_i, rebuilding the curve, repricing the trade being hedged, and measuring its price change. This method has the advantage of enabling very precise p/l explanation, at the cost of requiring repeated yield curve builds.

The resulting *instrument sensitivity* is closely related to the forward rate risk. To be precise, let the *response function* $\beta_i(T) \equiv \mathrm{d}F(T)/\mathrm{d}r_i$. Then, the instrument sensitivity is exactly $\beta_i^\wedge \delta_f$. Thus, response functions provide an ideal tool for examining curve build methods.

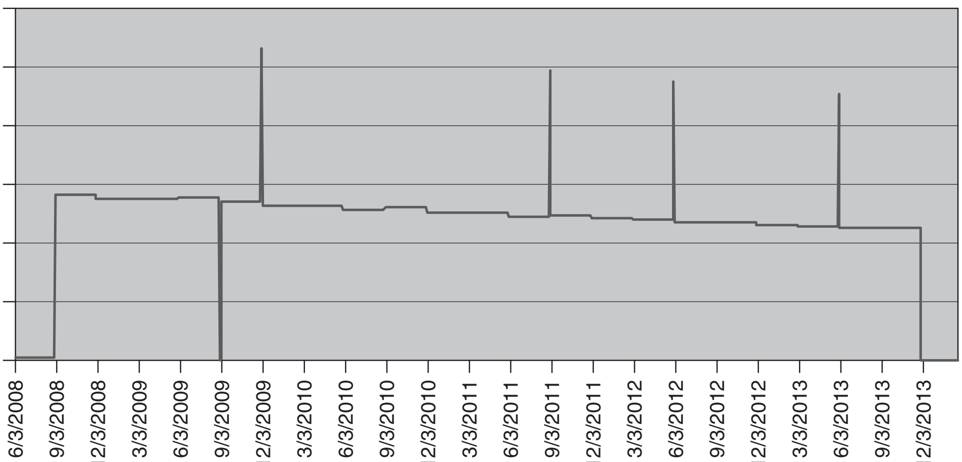

Figure 1 Forward rate risk for a (Payer) swap

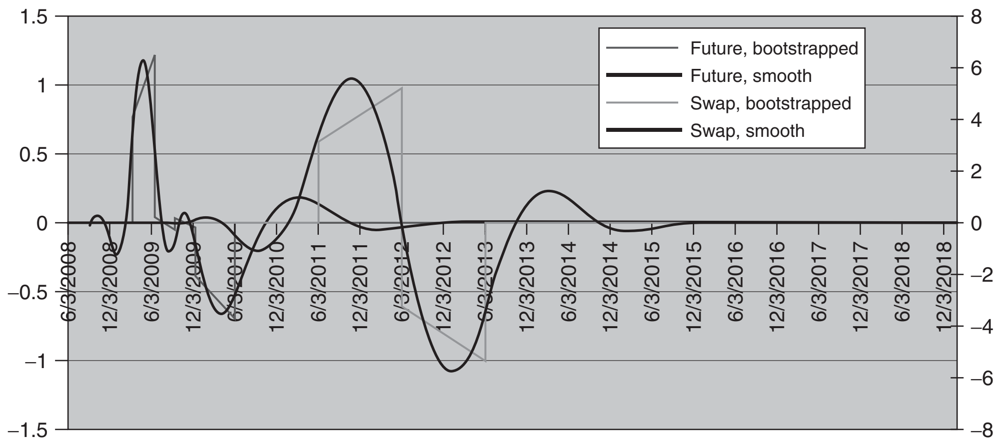

Figure 2 Response of F to fourth future and to 4-year swap

The response functions for two typical build instruments are displayed in Figure 2, for two different curve build methods. The response to a futures rate, shown against the left-hand scale, changes the forwards within the futures period and decreases them in the interval from the last future to the first swap (so that all other build instruments will have unchanged rates); naturally, within the futures period, $\mathrm{d}F(T)/\mathrm{d}t_i \simeq 1$. The response to a swap rate, which is substantially larger, is shown against the right-hand scale. In both cases, the bootstrapped curve shows the "sawteeth" characteristic of linear interpolation on y, while the smooth curve shows the inevitable loss of locality. This tension between smoothness and locality arises because a smooth curve, by its very nature, must alter values far from the source of a change in order to preserve smoothness; for details *see* **Yield Curve Construction**.

Bucket Delta Methods

Another common hedge method is to set the bucket end dates to the maturities of the curve build instruments, and then compute the *bucket deltas*: sensitivities $\vec{\delta}_k$ to the forward rate in the kth bucket, computed by applying a parallel shift to those forward rates. We can also define a Jacobian matrix J such that J_{ik} is the sensitivity of the ith instrument's implied rate to the forward rate in the kth bucket; then the instrument sensitivities are given by $J^{-1}\vec{\delta}$. This is known as the *inverse method*.

A hedge can also be constructed from $\vec{\delta}$ by minimizing the p/l variance of the hedge trade plus a portfolio of hedging instruments; for this we need an estimate of the covariance $\Sigma(T_k, T_m)$ between the forward rates $f(T_k)$ and $f(T_m)$. The variance is then a quadratic form in the hedge instrument notionals, which can easily be minimized. Any other quadratic form, such as a penalty function based on the hedge notionals, can be included without difficulty.

Nonlinear Products

For any product, the sensitivity δ_f is defined in each yield curve state; however, it need not be independent of that state. This nonlinearity is most pronounced for options, especially when they are short-dated and nearly at-the-money. In this case, to lock in an option value by hedging we must dynamically rebalance the hedging instruments, subject to the well-known limitations of payoff replication strategies.

One issue of particular importance is that the local hedge, based on δ_t in the current state of the yield curve, can differ greatly from the variance-minimizing hedge if the rebalancing frequency is finite or if jumps are present. This occurs when the distribution of possible curve shifts is strongly asymmetric, or more frequently when the second derivative of the payoff is highly state-dependent.

These issues are not unique to interest rates, but they can become more pronounced for some payoffs owing to the tendency of short rates to move in