Physica A 433 (2015) 171–181

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

# A quantitative description for efficient financial markets

Eero Immonen *

*Process Flow Solution Ltd, Kaurakatu 48B, 20740 Åbo, Finland*
*Financial Accounting and Optimization Systems at Åbo Akademi University, Henriksgatan 7, 20500 Åbo, Finland*

## HIGHLIGHTS

- A feedback control model for asset price dynamics and value discovery is presented.
- Market efficiency is defined as robust asymptotic price–value equality.
- A complete characterization of the trader structure for which the market is efficient is presented in the main result of the article.
- The main result illustrates that the more transparent the market is, the more efficient it is.
- Investor rationality is not required for the results to hold true.

## ARTICLE INFO

*Article history:*
Received 9 September 2014
Received in revised form 1 February 2015
Available online 1 April 2015

*Keywords:*
Efficient market hypothesis
Tâtonnenent
Internal model principle
Value discovery
Asset value dynamics

## ABSTRACT

In this article we develop a control system model for describing efficient financial markets. We define the efficiency of a financial market in quantitative terms by robust asymptotic price–value equality in this model. By invoking the Internal Model Principle of robust output regulation theory we then show that under No Bubble Conditions, in the proposed model, the market is efficient if and only if the following conditions hold true: (1) the traders, as a group, can identify any mispricing in asset value (even if no one single trader can do it accurately), and (2) the traders, as a group, incorporate an internal model of the value process (again, even if no one single trader knows it). This main result of the article, which deliberately avoids the requirement for investor rationality, demonstrates, in quantitative terms, that the more transparent the markets are, the more efficient they are. An extensive example is provided to illustrate the theoretical development.


## 1. Introduction

### 1.1. Background

One of the most remarkable empirical facts of capitalist economic systems is that they appear to allocate resources efficiently in the absence of any external guidance. The "invisible hand" theory, originally set forth by Adam Smith [1], postulates that if each consumer is allowed to freely choose what to purchase and each producer is allowed to freely choose his or her product line, the market will settle on a product distribution and prices that are beneficial for the entire economy. This settlement occurs by means of a self-regulating process, the Walrasian tâtonnement (see e.g. Ref. [2]), which involves a search of balance of net supply and demand for the products based on the current observed prices for the products.

That markets are capable of efficiently allocating resources and stabilizing the price of an asset implies that the markets must, in some sense, take into account all information affecting the assets price. This is the essential content of the celebrated

* Correspondence to: Financial Accounting and Optimization Systems at Åbo Akademi University, Henriksgatan 7, 20500 Åbo, Finland.
*E-mail address:* eeimmone@abo.fi.

http://dx.doi.org/10.1016/j.physa.2015.03.032

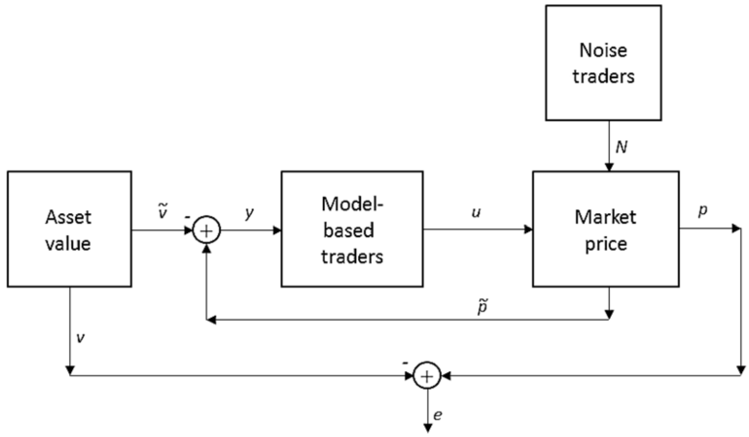


**Fig. 1.** The full value discovery process model for tâtonnement considered in the present article. Value discovery requires that $\lim_{t\to\infty} e(t) = 0$.

Efficient Market Hypothesis (EMH) due to Samuelson [3], Fama [4] and others. Although there exist many forms of the EMH, in broad terms they all assert that a market is efficient if prices immediately, for all practical purposes, reflect all relevant information about the assets on the market. The EMH thus requires that, on average, the population is always correct about the price (even if no single person is) and as new information appears, the market participants revise their expectations appropriately to maintain this state of affairs.

The degree to which the EMH holds true in practice has been debated in the academic literature over the course of decades (see Ref. [5] for a review). The observation that perfectly efficient asset prices imply purely random price fluctuations [3], and the subsequent conflicting rejection of the random walk property of observed asset prices [6], the existence of bubbles and crashes in asset prices [7], and the unusual profitability of simple technical strategies (see e.g. Ref. [8]), are among the key sources for criticism for the EMH. A conclusion of these studies is that the degree to which markets are efficient is likely not constant over time. In particular, as new information is being processed by market participants, there is a transient period (of unknown and varying duration) during which price may not reflect true value.

In spite of the progress made on understanding the nature of market efficiency, the actual mechanism by which prices adjust to new information – i.e. information processing by market participants during the tâtonnement process – appears to be relatively unknown [9]. In particular, to the author's knowledge there is no comprehensive mathematical model for price discovery based on the market participants' behavior. The purpose of this article is to fill this gap by presenting such a model for efficient markets. Our model explains, in rigorous terms, what it means for the markets to incorporate all available information about an asset, and provides necessary and sufficient conditions for value-based price discovery. The significance of such a model is not only in its ability to explain the behavior of market participants during tâtonnement but also, via the presented necessary conditions for EMH to hold, in the new directions it provides for testing – and perhaps rejecting – the hypothesis in practice.

### 1.2. Contribution of this article

The model developed in Section 2 allows us to formulate the entire Walrasian tâtonnement process as a robust output regulation problem. We can then invoke the celebrated Internal Model Principle of control theory (see e.g. Refs. [10,11]) to establish the main result of the article in Theorem 1. It shows that under No Bubble Conditions, the market for an asset $\mathcal{A}$ is efficient precisely when the following two conditions hold true:

1. The traders, as a group, can identify any mispricing in asset $\mathcal{A}$;
2. The traders, as a group, incorporate an internal model of the value process for $\mathcal{A}$.

A remarkable feature of our model is that, besides linear deterministicity discussed below, we make relatively few assumptions about the specific structure of the markets and about the arrangement of the individual traders and investors. As demonstrated in Fig. 1, we essentially treat the investors and the market as interconnected "black boxes" whose dynamical properties result from the interaction of – potentially a vast number of – individuals. Consequently, our modeling framework can simultaneously incorporate any number of traders with different trading strategies. These can include, among others, arbitrage strategies, value-driven ones whereby the traders' actions are driven by a perceived price–value discrepancy, and momentum-based ones, depending on positive feedback, without any regard to the specific design of their individual trading strategies. Further, it is remarkable that the occurrence of price–value discovery in our model does not depend on all market participants being rational. Indeed, part of the net demand–supply affecting the current price level in the presented model results explicitly from potentially irrational investors. In addition to this, our modeling approach to the market place dynamics allows us to incorporate a number of typical market microstructure models, such as the "law of the market" considered by Mosetti [12].

In this article, all dynamical systems are deterministic, linear and finite-dimensional. We conclude this subsection by addressing these individual choices separately.

The assumption of *deterministicity* is justified by our formulation of the value discovery process as a *robust* output regulation problem: In our framework, the trader model and the market model are a priori known to be uncertain and subject to additive perturbations and/or parameter drift. We require asymptotic price–value equality irrespective of what the true value is and irrespective of these uncertainties, which are not necessarily small in magnitude in our model. Taking stochastic uncertainty into account is a topic for future research.

Our standing assumption of *linearity*, on the other hand, covers the transient period *after* a "news shock", which leads to value discovery under the given conditions. We do not assume system linearity *at* the time of news arrival. Indeed, traders may revise their linear models in nonlinear ways at the time of such news shocks; our robustness results provide conditions for market efficiency to prevail. Further, system linearity during the transient period is motivated by the observation that a linear approximation of an underlying nonlinear process is often accurate at least locally, in the vicinity of the present operating point, which in this case is the current state of a market (and traders) after a "news shock". While the reader may consider linearity to be a strong assumption, we point out that nonlinearity, in an of itself, would be a rather general assumption in this context. Indeed, the Internal Model Principle, which is provides an equivalence relation between the controller structure and solvability of a regulation problem, and which is the foundation of our approach, has only been proved for linear systems [10,11]. Significant progress has been made on nonlinear systems, too [13,14]. Consequently, an important topic for future research is to establish a suitable nonlinear systems framework, covering typical nonlinear models for financial markets, for which the results of this article can be generalized.

Finally, while today several generalizations of the Internal Model Principle exist for *infinite-dimensional* systems (see e.g. Ref. [11]), none of them appears to address the case in which the exosystem output is *not* directly observed by the controller. This is a crucial feature in our model—individual market participants only know their individual *estimates* of true value.

### *1.3. Relation to prior work*

Conditions for the existence and uniqueness of a stable price, i.e. the end result of the tâtonnement process—are provided by the well-known general equilibrium models of economics (see e.g. Ref. [15] and the references therein). The transient state, i.e. information processing by the market participants, preceding the equilibrium state, however, is much less well understood although significant progress has been made recently. Topics such as the path of price discovery [16], the speed of convergence of price to an equilibrium [17], the market impact of changes in supply and demand [15], and the effect of specific trading strategies on price discovery [18,19], have received considerable attention in the recent academic literature.

In this article, we do not attempt to show how agent-specific optimization leads to value discovery. In contrast, we focus our attention on the necessary and sufficient feedback structure of the market system, for which price efficiency is achievable. In practice, then, the aggregate effect of agent-specific optimization should be the fulfilment of these conditions.

Several researchers have utilized dynamic systems and feedback control theory for describing trading and the tâtonnement process. By interpreting trading strategies as a feedback controller, Alvarez-Ramirez et al. [18] showed that trend followers can lead to oscillatory phenomena, and that adaptation mechanisms driven by fundamental value considerations are necessary in order for prices to track values. Farmer and Joshi [19] studied the price dynamics induced by several commonly used (deterministic) trading strategies, including value investing and trend following. Hommes [20] observed that the aggregate effect of simple traders at the micro level may be a sophisticated structure at the macro level. More recently, Chiarella et al. [21] estimated the parameters of a dynamical system consisting of fundamental and chartist traders; their model could explain, for example, the inflation and deflation of bubbles.

We believe this article presents the first formulation of the entire the value discovery process as a generic interconnection of linear dynamical systems (Fig. 1). The results we derive in this modeling framework thus differ from those cited above in that we do not need to impose any explicit specification for traders or the market mechanism transforming supply and demand to a price–value. Further, the main result of this article, which provides necessary and sufficient conditions for market efficiency, appears to be new: It is the first quantitative description connecting market efficiency directly to the information that the traders must (and need not) possess, and to the feedback structure of the traders as a group. Its formulation and proof involve an adaptation of the well-known Internal Model Principle of control theory (see e.g. Refs. [10,11]) to the financial markets' situation.

To our knowledge, only Mosetti [12] has attempted to provide a conceptual description of the structure of a market with stable prices using the Internal Model Principle, albeit without direct reference to the EMH. Mosetti's results assume a perfect foresight of the equilibrium prices, whereas in this article no one single trader has to know the true value. Further, in Mosetti's results [12], supply and demand are driven by expectations of future prices, without regard to how such expectations are formed. In contrast, in this paper we specify the necessary and sufficient structure of traders leading to "stable" prices in the sense of market efficiency. Finally, while the price process is driven by a regulator system in Mosetti's work [12], there appears to be no clear link between market participants and the regulator. In particular, Mosetti does not address the possible dynamical specifications of traders' trading strategies, nor consider uncertainty within them. In our approach, the role of certain traders as a regulator is clearly described; here the traders (as a group) only learn the value, i.e. the current equilibrium price, of the asset. Finally, our model can be subject to unknown (additive) perturbations to the model's parameters, as well as disturbance supply–demand generated by noise traders.

## 2. The value discovery process model

In this section we develop a feedback control system model for Walrasian tâtonnement, which we call the value discovery process model, as illustrated in Fig. 1. We hasten to emphasize that, while the development presented below may at the outset seem abstract – and perhaps even contrived – it includes many of the models presented in the academic literature on finance an economics as special cases. Our model covers asset value dynamics, market price dynamics and trader dynamics, which are discussed in separate subsections below. Note that throughout this article we can assume logarithmic prices, to allow for negative values.

### 2.1. Asset value dynamics

Let $\mathcal{W}$ be a finite-dimensional vector space. The value, and traders' estimates thereof, of $\mathcal{A}$ are then assumed to be described by the following linear dynamical system:

$$\dot{w}(t) = Sw(t), \qquad w(0) = w_0 \in \mathcal{W} \tag{1a}$$

$$v(t) = Qw(t), \quad \forall t \geq 0 \tag{1b}$$

$$\tilde{v}(t) = \tilde{Q}w(t), \quad \forall t \geq 0 \tag{1c}$$

where the linear maps $S : \mathcal{W} \to \mathcal{W}$, $Q : \mathcal{W} \to \mathbb{R}$ and $\tilde{Q} : \mathcal{W} \to \mathcal{Y}$, where $\mathcal{Y}$ is another (finite-dimensional) linear space.

The linear space $\mathcal{W}$ can be thought to consist of all factors that affect the value of asset $\mathcal{A}$ from the signal generation point of view [11]. In Eq. (1b) $v(t)$ denotes the *true* value of asset $\mathcal{A}$ (e.g. after a news shock at $t = 0$), which is a scalar function $v(t) = Q e^{St} w_0$, $t \geq 0$. It is important to observe that while we assume the existence of a "true" value, it is not necessarily known to any one trader. Instead, by Eq. (1c) traders of asset $\mathcal{A}$ have formulated their *estimates* of value, which are collected in the vector $\tilde{v}(t) = \tilde{Q} e^{St} w_0$, $t \geq 0$.

In our model, any news event can be regarded as a change of parameters in the above value process, thus starting a new tâtonnement process. We can therefore, without loss of generality, assume that during the tâtonnement process, there are no additional news events affecting the value process (for otherwise the process would start over again).

To see how the dynamical system (1) relates to the well-known asset valuation models of contemporary financial theory, consider the Capital Asset Pricing Model (CAPM) equation:

$$r_{\mathcal{A}} = \alpha_{\mathcal{A}} + \beta_{\mathcal{A}} \cdot MRF \tag{2}$$

where $\alpha_A$ and $\beta_A$ are asset-specific parameters obtained from regression analysis, while *MRF* denotes the expected market return in excess of the risk-free rate. If we interpret the CAPM equation (2) as a prediction of the constant future value for asset $\mathcal{A}$, then it is easy to specify an equivalent system (1) to generate this value. To this end let the current market price of $\mathcal{A}$ be $P_A$. Then, assuming logarithmic returns, we can choose $\mathcal{W} = \mathbb{R}^2$, $S = 0 \in \mathbb{R}^{2\times 2}$, $Q = \begin{pmatrix} \alpha_A + \log P_{\mathcal{A}} & \beta_{\mathcal{A}} \end{pmatrix}$ and $w_0 = \begin{pmatrix} 1 & MRF \end{pmatrix}^T$. Clearly this specification yields for $\mathcal{A}$ the true value $v(t) = Q e^{St} w_0 = Q w_0 = \log P_{\mathcal{A}} + \alpha_{\mathcal{A}} + \beta_{\mathcal{A}} \cdot MRF$ for all $t \geq 0$, which is just a restatement of Eq. (2). We stress that, in our framework, the actual values of the coefficients $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$ – and hence also the true value $v$ – may be unknown to the traders. Also note that, in this example, any news event concerning stock $\mathcal{A}$ can be assumed to affect the true value of $A$ through a change in $\alpha_{\mathcal{A}}$ or $\beta_{\mathcal{A}}$, or even through a change in *MRF* if the news is significant enough.

The system (1) can also be chosen to generate seasonal (periodic) values, ramps and other typical signals. For more information, we refer the reader to the author's thesis [11]. The simplest dynamical system (1) for generation of constant signals is utilized in Section 4 .

### 2.2. Market price dynamics

Let $\mathcal{U}$ be a finite-dimensional vector space. The net demand–supply from so-called model-based traders (see Section 2.3.1) is denoted by $u(t) \in \mathcal{U}$ for each $t \geq 0$. In the simplest case $\mathcal{U} = \mathbb{R}$, whereby $u(t) > 0$ (resp. $u(t) < 0$) signals that there is, in total, more buying than selling (resp. more selling than buying) of $\mathcal{A}$, and the price of $\mathcal{A}$ should tend to rise (resp. fall). However, as we shall see in Section 4, it is convenient to allow for vector-valued functions $u$.

The dynamical system, which transforms the aggregate net demand–supply to the asset price $p(t) \in \mathbb{R}$, is defined as follows:

$$\dot{z}(t) = Az(t) + Bu(t) + N(t), \qquad z(0) = z_0 \in \mathcal{Z} \tag{3a}$$

$$p(t) = Cz(t), \quad \forall t \geq 0 \tag{3b}$$

where $\mathcal{Z}$ is a finite-dimensional vector space and where the linear maps $A : \mathcal{Z} \to \mathcal{Z}$, $B : \mathcal{U} \to \mathcal{Z}$ and $C : \mathcal{Z} \to \mathbb{R}$. In Eq. (3a) $N(t)$ denotes the effect of noise trading, as defined in Section 2.3.2.

The matrices $A$, $B$ and $C$ in system (3) depend on the chosen market microstructure model. For example, the "law of the market" considered by Mosetti [12] is specified as $\dot{p} = H(u)$, where $H(\cdot)$ is a function such that $H(0) = 0$ and $H' > 0$, and where $u(\cdot)$ is an excess demand function. By linearizing this differential equation, we obtain an equivalent (local) realization (3) as follows: $\mathcal{Z} = \mathbb{R}$, $A = 0$, $B = \tilde{H} > 0$ (constant from linearization), and $C = 1$.

### 2.3. Trader dynamics

In our modeling framework, there are two categories of traders for asset $\mathcal{A}$; namely those who base their trading decisions on some form of market analysis for asset $\mathcal{A}$ and those whose net demand–supply for $\mathcal{A}$ is independent of the price and value of $\mathcal{A}$. These traders are referred to as *model-based traders* and *noise traders*, respectively, and they are defined in the following Subsections.

#### 2.3.1. Model-based traders

Referring to Fig. 1, the model-based traders (as a group) are assumed utilize an interpretation $\tilde{p}(\cdot)$ of the state of the market together with their estimate of value $\tilde{v}(\cdot)$, to come up with their aggregate net demand–supply $u(\cdot)$. As indicated in Fig. 1, this interpretation of market price is not necessarily equal to the asset price $p(\cdot)$ in (3b), but in simple cases we may have $\tilde{p} = p$. Further, the value estimate $\tilde{v}$ is not necessarily equal to the true value $v$ of asset $\mathcal{A}$, which may even be unknown to the traders.

To describe the dynamics of $u(t) \in \mathcal{U}$ for *all* such model-based traders, let $\mathcal{X}$ be a finite-dimensional vector space and consider the following linear dynamical system:

$$\dot{x}(t) = A_c x(t) + B_c y(t), \qquad x(0) = x_0 \in \mathcal{X} \tag{4a}$$

$$u(t) = C_c x(t) + D_c y(t), \quad \forall t \geq 0 \tag{4b}$$

$$y(t) = \tilde{C} z(t) - \tilde{Q} w(t) = \tilde{p}(t) - \tilde{v}(t), \quad \forall t \geq 0 \tag{4c}$$

where the linear maps $A_c : \mathcal{X} \to \mathcal{X}$, $B_c : \mathcal{U} \to \mathcal{X}$, $C_c : \mathcal{X} \to \mathcal{U}$, $D_c : \mathcal{Y} \to \mathcal{U}$ and $\tilde{C} : \mathcal{Z} \to \mathcal{Y}$.

The essence of $y$, which drives Eqs. (4a) and (4b), and which is defined in (4c), is as follows: Model-based traders make an *interpretation* $\tilde{C} z(t) = \tilde{p}(t)$ of the state of the market at time $t$, and they compare it to their present estimate of "fair value" $\tilde{v}(t) = \tilde{Q} w(t)$ of this quantity. The simplest case is $\tilde{C} = \mathbf{1} C$ and $\tilde{Q} = Q$, whereby all model-based traders are assumed to compare the observed market price directly to the true asset value. However, as we shall show in Section 4, this definition can also cover, among others, arbitrage traders whose induced net demand–supply only depends on the observed price difference between two (or more) exchanges. It is also important to emphasize that, in system (4), the traders' decisions do not need to be independent of each other.

Farmer and Joshi [19] considered "fundamental value" traders acting upon the feedback law $u(t) = \lambda y(t) = \lambda(p(t) - v(t))$. A realization (4) for this law is clearly obtained by setting $\mathcal{X} = \mathbb{R}$, $A_c = B_c = C_c = 0$ and $D_c = \lambda$ in Eqs. (4), and by assuming that $\tilde{C} = C$ and $\tilde{Q} = Q$. On the other hand, Alvarez-Ramirez et al. (cf. Ref. [18, Result 1]) considered traders utilizing a proportional–integral feedback law of the market price and a (constant) price forecast $v$:

$$u(t) = k_p(p(t) - v) + k_i \int_0^t (p(\sigma) - v)\mathrm{d}\sigma. \tag{5}$$

According to Alvarez-Ramirez et al. [18], a trend following (resp. contrarian) trading strategy corresponds to $k_p > 0$ and $k_i > 0$ (resp. $k_p < 0$ and $k_i < 0$). We can easily build a state space realization (4) for this trading strategy as follows: Choose $\mathcal{X} = \mathbb{R}$, $A_c = 0$, $B_c = k_i$, $C_c = 1$, $D_c = k_p$. In Ref. [18] the traders had to utilize the true asset value $v$, while we will demonstrate in Section 4 that traders can be arranged so that true value needs not be known to any one trader.

#### 2.3.2. Noise traders

In our modeling framework, noise traders act as a disturbance to the net demand–supply for asset $\mathcal{A}$. They are assumed to act without regard to actual asset values (or any estimates thereof). Thus we may specify their effect $N(t)$ as follows:

$$N(t) = P w(t), \quad \forall t \geq 0 \tag{6}$$

where the linear map $P : \mathcal{W} \to \mathcal{Z}$ is typically unknown. It should therefore be emphasized that Eq. (6) is generic in the sense that it only imposes a restriction on the *dynamical* behavior of the noise traders. Also note that if we can decompose $P = BP_n$ for some linear map $P_n : \mathcal{W} \to \mathcal{U}$, then we can decompose the net demand–supply in (3a) as $u(t) = u_m(t) + u_n(t)$ where $u_m(t)$ is the net demand–supply from model-based traders and $u_n(t)$ is a net demand–supply from noise traders. In this special case noise traders affect the price in the same way as model-based traders, but the specification (6) also allows for more general situations.

As an example, the effect of forced liquidations by funds holding asset $\mathcal{A}$ can be modeled using Eq. (6). Noise traders can thus be regarded as (potentially) acting irrationally.

## 3. A quantitative description for efficient markets

Intuitively speaking, the market system presented in Section 2 is "efficient" whenever the market price $p(t)$ in Eq. (3b) reaches the asset's value $v(t)$ in Eq. (1b) at least asymptotically, i.e. as $t \to \infty$. In Fig. 1 this is seen as $e(t) \to 0$ as $t \to \infty$. Our concern in this section is to define market efficiency precisely in the modeling framework of Section 2 and to prove necessary and sufficient conditions for it, by invoking the Internal Model Principle of robust control theory.

Following the article [10] of Francis and Wonham, we shall, without any loss of generality, make the following standing assumptions throughout the remainder of this article. See Ref. [10] and the references for justification of these assumptions.

1. All eigenvalues of $S$ have a nonnegative real part.
2. $\mathbf{R}(\tilde{C}) + \mathbf{R}(\tilde{Q}) = \mathcal{Y}$, where $\mathbf{R}$ denotes the range of a linear map.
3. $C \neq 0$.

We begin by defining a number important concepts. The following definition is adapted from the concept of readability in Francis and Wonham [10].

**Definition 1.** We say that the traders, as a group, can identify any mispricing in asset $\mathcal{A}$ if there exists a linear map $K : \mathcal{Y} \to \mathbb{R}$ such that

$$C = K\tilde{C}, \quad \text{and} \quad Q = K\tilde{Q}. \tag{7}$$

Any linear map $K : \mathcal{Y} \to \mathbb{R}$ satisfying these two conditions is referred to as a readability map.

Definition 1 simply states that the information utilized by traders (as a group) is sufficient for inferring the true asset value from it by $v(t) = Qw(t) = K\tilde{Q}w(t)$, for all $t \geq 0$, and that the traders (as a group) are able to compare it to the present asset price $p(t) = Cz(t) = K\tilde{C}z(t)$. Thus, the traders (as a group) can identify any nonzero value of $v(t) - p(t)$, i.e. a mispricing in $\mathcal{A}$. We emphasize that, in Definition 1, no one single trader is required to know the matrix $K$; only its existence is required.

The following two definitions are also adapted from Refs. [10,11] to our framework:

**Definition 2** (*No Bubble Condition*)**.** If the closed loop matrix $A_L$, defined by

$$A_L = \begin{pmatrix} A + BD_c\tilde{C} & BC_c \\ B_c\tilde{C} & A_c \end{pmatrix} \tag{8}$$

is stable, i.e. all eigenvalues have negative real parts, then we say that there is no bubble in the pricing of asset $\mathcal{A}$.

The relation of loop stability to asset bubbles in our terminology can be explained as follows. If the No Bubble Condition (NBC) above holds, then the asset price $p(\cdot)$ is uniformly bounded for all uniformly bounded inputs to the loop system. In particular, under the NBC, the traders cannot make the price rise exponentially (as in market bubbles) or fall exponentially (as in market crashes), beyond all bounds, whenever their collective interpretation, i.e. $y$, of asset mispricing is bounded. Furthermore, the NBC guarantees that there are no internal system instabilities, which are currently transparent to the market price but which may show up on a small change of system parameters. In practice these could be e.g. programming errors in the market's order matching system.

In practice, the NBC is verified by checking the eigenvalues of the closed loop system. On the other hand, matrices $A_c$, $B_c$, $C_c$ and $D_c$ satisfying the NBC can be found by using standard stabilization techniques of linear systems theory (see e.g. Ref. [22] and the references therein).

**Definition 3** (*Robustness*)**.** Let $\Pi$ be a property of the closed loop system, consisting of (1), (3), (4) and (6). We say that $\Pi$ is robust with respect to a set $\Omega$ of parameters if $\Pi$ holds in an open neighborhood of $\Omega$, as the parameters are subjected to additive perturbations.

Since the eigenvalues of a matrix depend continuously on the entries of that matrix, if $A_L$ is stable, then $A_L$ remains stable in the presence of small additive perturbations to $\Omega_{A_L} = \{A, B, C, A_c, B_c, C_c, D_c, \tilde{C}\}$. Thus the NBC is robust with respect to $\Omega_{A_L}$.

To study market efficiency, we still need to define the concept of an internal model. To do this, we choose to slightly modify the operator-theoretic definition introduced in Ref. [11] since it requires no additional concepts. We point out, however, that several earlier analogues are available for finite-dimensional systems, see e.g. Francis and Wonham [10].

**Definition 4** (*Internal Model*)**.** Let $K$ be any readability map and let $\Lambda : \mathcal{W} \to \mathcal{X}$ and $\Delta : \mathcal{W} \to \mathcal{Y}$ be arbitrary matrices. We say that the traders, as a group, incorporate an internal model of the value process whenever the following relation holds true:

$$\Lambda S = A_c\Lambda + B_c\Delta \Longrightarrow K\Delta = 0. \tag{9}$$

If $K = I$, the identity matrix, then relation (9) reduces to that introduced in Ref. [23]. This can be the case, for example, if the model-based traders know the true value and compare the actual price to it ($y = e$).

If the NBC holds, if $\mathcal{Y} = \mathbb{R}$, $K = I$, and if the traders incorporate an internal model of the value process, then it can be shown (see e.g. Ref. [11, Theorem 6.20]) that there exists a matrix $\Lambda$ such that $\Lambda S = A_c\Lambda$. Hence in this case the interpretation of Definition 4 is that there exists an $A_c$-invariant subspace, namely $\mathbf{R}(\Lambda) \subset \mathcal{X}$, where the dynamics $A_c$ can be described by $S$ [11]. From the signal generation point of view this means that any signal generated by the value process can be reproduced by the traders (as a group) [11].

**Definition 5** (*Efficient Market*)**.** In the closed loop system, consisting of (1), (3), (4) and (6), we say that the market (for asset $\mathcal{A}$) is efficient if the following conditions are satisfied:

1. For each $w(0)$ in $\mathcal{W}$, $x(0) \in \mathcal{X}$ and $z(0) \in \mathcal{Z}$, we have $|p(t) - v(t)| \leq Me^{-at}$ for some constants $M > 0$ and $a > 0$, such that $a$ is independent of $w(0)$, $x(0)$ and $z(0)$;
2. Condition 1 is robust with respect to $\Omega = \{A, B, C, C_c, D_c, P, \tilde{C}, \tilde{Q}, Q\}$.

It is important to observe that our formulation of market efficiency above does not imply rational behavior for the traders. On the contrary, if market efficiency holds in the sense of Definition 5, then $|p(t) - v(t)| \to 0$ at an exponential rate even in the presence of irrational traders (Condition 1), whose behavior, i.e. matrix $P$, can change within the scope of our closed loop model (Condition 2). Another important point to observe is that we specifically avoid uncertainty in the internal model (i.e. matrices $A_c$ and $B_c$).

We conclude this section with the main result of the present article, followed by two clarifying remarks. The proof of this main result is given in the Appendix.

**Theorem 1.** *Assume that the No Bubble Condition holds for the closed loop system consisting of* (1), (3), (4) *and* (6)*. Then the market for asset $\mathcal{A}$ is efficient, such that $P$ and $\tilde{Q}$ can be chosen arbitrarily, if and only if the following two conditions hold true:*

1. *The traders, as a group, can identify any mispricing in asset $\mathcal{A}$, and condition* (7) *holds for the perturbed matrices for some, possibly also perturbed, readability map $K$;*
2. *The traders, as a group, incorporate an internal model of the value process* (1)*.*

**Remark 1.** Condition 1 in Theorem 1 specifies the linear constraints (7) for perturbations to $\tilde{Q}$, $Q$, $\tilde{C}$ and $C$. Consequently, even if $\tilde{Q}$ is arbitrary, the other matrices $Q$, $\tilde{C}$ and $C$ are, in general, not arbitrary. However, besides these constraints, there are no additional restrictions on perturbations to $\tilde{Q}$ and $Q$ (such as small norm). In the special case where $e = y$, i.e. $\tilde{Q} = Q$, we can set $K = I$, so that both $P$ and $Q$ can be chosen arbitrarily. In this case, the perturbations to $C$ can be independent of those to $Q$ [23].

**Remark 2.** Since Theorem 1 is valid under the NBC, all perturbations to the system parameters must retain the NBC.

The following section presents an example of trader arrangement satisfying the two conditions in Theorem 1.

## 4. Example—an asset traded on two markets

In this section we shall present an example to illustrate the theoretical development of the previous sections. Fig. 2 displays the results of MATLAB simulations[1] for this example. Each panel in Fig. 2 shows two asset prices $p_1$ and $p_2$, defined in Eq. (10) below, as solid lines. The single dashed line represents the current asset value, where applicable.

As a generalization of the simple "law of the market" [12], in this section we shall consider the following two-exchange market model (3) for the price $p(t)$ of asset $\mathcal{A}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} = \begin{pmatrix} -a_1 & a_1 \\ a_2 & -a_2 \end{pmatrix}\begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} + \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + \begin{pmatrix} N_1(t) \\ N_2(t) \end{pmatrix} \tag{10a}$$

$$p(t) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}, \quad t \geq 0 \tag{10b}$$

with $p_1(0), p_2(0) \in \mathbb{R}$. Here $p_1(t)$ and $p_2(t)$ denote the prices of $\mathcal{A}$ at exchanges 1 and 2, respectively, at time $t$. By Eq. (10b), the true price $p(t)$ of $\mathcal{A}$ is defined as the average of them. The effect of noise traders to the two exchanges are denoted by $N_1$ and $N_2$.

The parameters $a_1 > 0$ and $a_2 > 0$ in Eqs. (10) can be used to model arbitrage trading and open orders existing at the exchange: If (say) $p_1(t_0) < p_2(t_0)$, for some $t_0 \geq 0$, then $p_2(t)$ tends to decrease and $p_1(t)$ tends to increase for $t > t_0$, as long as there is no net demand–supply (i.e. $u_1 \equiv u_2 \equiv 0$). The speed at which this occurs depends on $a_1$ and $a_2$. Fig. 2 (Top Left Panel) illustrates this convergence of market prices. Note that if either $u_i$, $i = 1, 2$, is not uniformly zero, then there is no intrinsic guarantee that $|p_1(t) - p_2(t)| \to 0$ as $t \to \infty$; we need arbitrage traders' input to achieve this (see below).

The parameters $b_1 > 0$ and $b_2 > 0$ in Eqs. (10), on the other hand, depend on the liquidity of $\mathcal{A}$: Since $u_i(t)$ is the net demand–supply for $\mathcal{A}$ at exchange $i$ at time $t$, a larger $b_i$ implies a larger change in price for a given fixed nonzero $u_i(t)$.

The market model (10) will naturally result in a constant price, i.e. $p(t) \to p_\infty \in \mathbb{R}$ as $t \to \infty$, whenever there is no net demand–supply. This system is therefore unstable in the sense that any bounded nonzero net demand ($u_1 > 0$ and $u_2 > 0$) will result in an unbounded asset price as $t \to \infty$. The model-based traders, defined below, must act to stabilize system, for otherwise a bounded value reference may lead to an unbounded asset price—an asset value bubble.

[1] The corresponding MATLAB code is available from the author.

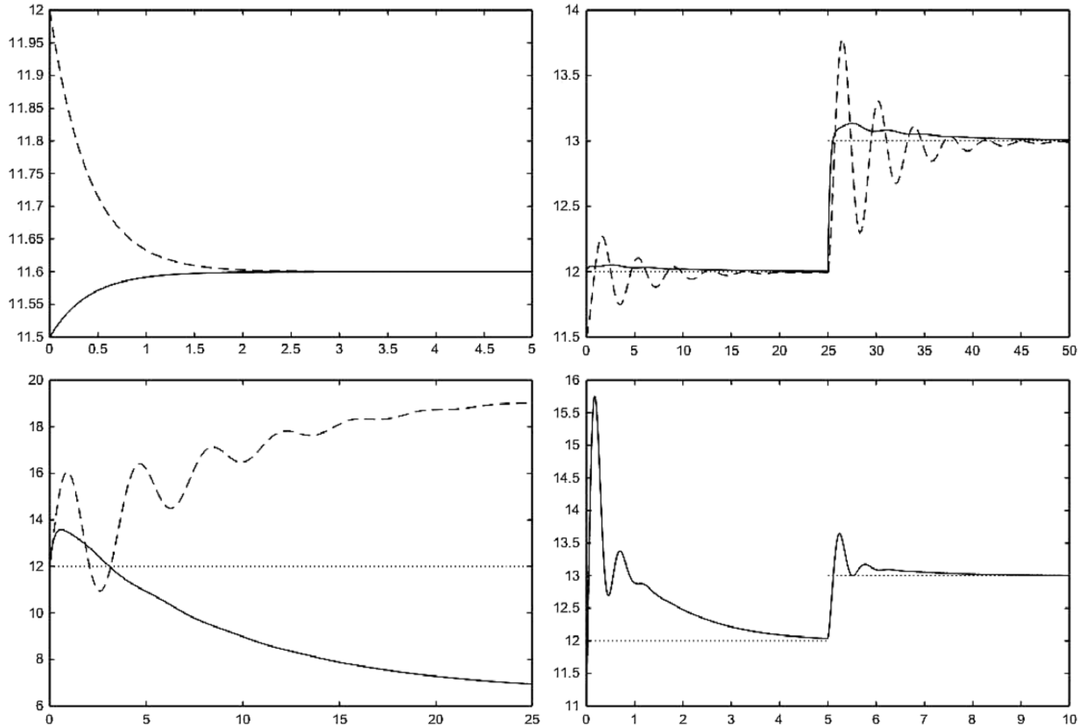

**Fig. 2.** Top left: Market model in the absence of inputs—prices at the two exchanges converge. Top right: Simple feedback traders (13) can adapt to changes in the known value. Bottom left: Simple feedback traders (13) with unknown value and noise trading (divergence). Bottom right: An efficient market—price tracks value even if it is unknown, and even if there are model uncertainties and noise traders.

Let us first consider a scenario where there are only two traders; one trading at each of the exchanges. We impose the following specification (4) for these traders (this analogous to the technical traders discussed in Ref. [18]):

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} a_{11}^c & a_{12}^c \\ a_{21}^c & a_{22}^c \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} k_1^i & 0 \\ 0 & k_2^i \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \tag{11a}$$

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} k_1^p & 0 \\ 0 & k_2^p \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \tag{11b}$$

$$y_j(t) = p(t) - v(t), \quad j \in \{1, 2\}. \tag{11c}$$

Here the value $v$ is assumed to be known. If each $a_{ij}^c = 0$, then Eqs. (11) reduce to the technical traders in Ref. [18, Result 1]. However, nonzero values for $a_{ij}^c$ allow us to model trader learning and herding, i.e. interdependence of the two traders' decisions.

We shall consider the following simple dynamical system for asset value generation:

$$\dot{w}(t) = 0, \quad t \geq 0, \qquad w(0) \in \mathbb{R} \tag{12a}$$

$$v(t) = qw(t), \quad t \geq 0, \ q > 0. \tag{12b}$$

Clearly any constant value $v$ can be generated by using the system (12), and this is (isomorphic to) the simplest possible dynamical system for generating any constant signal [11]. Initially we assume that, in Eq. (12b), $q$ is known, but that assumption will be relaxed later.

For the sake of a numerical example, let us specify $a_{11}^c = a_{12}^c = a_{22}^c = k_2^i = k_1^p = 0$, $a_c21 = 0.3$, $k_1^i = -2$, $k_p^2 = -1$. Then Trader 1 employs a value strategy taking into account the past history [18], and Trader 2 is only considering the current price–value relationship but adapts his/her decisions to those of Trader1:

$$u_1(t) = c_1 - 2\int_0^t (p(s) - v(s))\mathrm{d}s, \quad t \geq 0, \ c_1 \in \mathbb{R} \tag{13a}$$

$$u_2(t) = c_2 + 0.3\int_0^t u_1(s)\mathrm{d}s - (p(t) - v(t)), \quad t \geq 0, \ c_2 \in \mathbb{R}. \tag{13b}$$

It is easy to verify that this arrangement results in a stable closed loop system. Furthermore, if there are no noise traders, i.e. if $N_1(t) = N_2(t) = 0$ for each $t \geq 0$, then the market price (10b) automatically adapts to each change of value during a tâtonnement process; see Fig. 2 (Top Right Panel). However, this market is not efficient in the sense of Definition 5. In fact, we only have to introduce noise traders via a nonzero $N(t) = Pw(t)$, with $P = \begin{pmatrix} 0.5 & 0.3 \end{pmatrix}^T$. The resulting price trajectories at the two exchanges for a constant $v(t)$ are shown in Fig. 2 (Bottom Left Panel). Clearly their average differs from the value $v(t)$.

We will conclude this example by showing a scenario where this market becomes efficient in the sense of Definition 5. Indeed, for the trader arrangement presented below, the price tracks any value as $|p(t) - v(t)| < Me^{-at}$ for some $M, a > 0$ and all $t \geq 0$ regardless of the initial state of the market and the model-based traders. Furthermore, this occurs in the presence of arbitrary noise traders (i.e. for any $P \in \mathbb{R}^2$), under uncertainty in the market model parameters (10), and in the absence of accurate knowledge of true value (12b). The traders will be arranged such that they, as a group, attempt to estimate the current state of the following system describing the interaction of the market (10) and the noise traders:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} z(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} A & P \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u(t) \tag{14a}$$

$$y(t) = \begin{pmatrix} C & -Q \end{pmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} \tag{14b}$$

where $A$, $B$, $C$, $P$ and $Q$ can be uncertain. Let $\Pi$ and $\Gamma$ be matrices satisfying the so-called regulator equations (cf. Ref. [11]) $A\Pi + B\Gamma + P_0 = 0$ and $C\Pi - Q_0 = 0$ for some nonzero $P_0, Q_0$. Then choose matrices $K$ and $L = \begin{pmatrix} L_1 & L_2 \end{pmatrix}^T$ such that $L_2 \in \mathbb{R}$ is nonzero, $A + BK$ is stable and such that $\begin{pmatrix} A & P_0 \\ 0 & 0 \end{pmatrix} - L\begin{pmatrix} C & -Q_0 \end{pmatrix}$ is stable. Finally specify $A_c$, $B_c$, $C_c$ and $D_c$ as follows:

$$A_c = \begin{pmatrix} A + BK - L_1C & P_0 + B(\Gamma - K\Pi) + L_1Q_0 \\ -L_2C & L_2Q_0 \end{pmatrix}, \qquad B_c = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \tag{15a}$$

$$C_c = \begin{pmatrix} K & \Gamma - K\Pi \end{pmatrix}, \qquad D_c = 0. \tag{15b}$$

It can be shown (see e.g. Ref. [11, Theorem 4.15]) that the closed loop system matrix (8) is in this case similar to an upper triangular block matrix with $A + BK$ and $\begin{pmatrix} A & P_0 \\ 0 & 0 \end{pmatrix} - L\begin{pmatrix} C & -Q_0 \end{pmatrix}$ on the diagonal. Thus the closed loop is stable. Further, it can be shown using the technique of proof of Lemma 6.52 in Ref. [11] that the traders, as a group, incorporate an internal model of the value process (12). Finally, as long as $\tilde{C}$ and $\tilde{Q}$ are chosen such that the traders can identify any mispricing in the asset value (e.g. $\tilde{C} = rC$ and $\tilde{Q} = rQ$ for some nonzero $r \in \mathbb{R}$ which need not be known), the conditions of Theorem 1 hold. Fig. 2 (Bottom Right Panel) shows the trajectory of a sample simulation whereby price tracks value in the presence of noise and parameter uncertainty.

One should bear in mind that the above trader specification (15) alone does not guarantee that the prices would converge, i.e. $|p_1(t) - p_2(t)| \to 0$ as $t \to \infty$, even if their average tracks any value as demonstrated above. Naturally, one can augment the trader model (15) with additional arbitrage traders driven by $y(t) = p_1(t) - p_2(t)$, such that their corresponding value reference is zero, to accomplish this.

## 5. Discussion, conclusions and suggestions for future work

In this article we have developed a feedback control model for efficient financial markets. We have shown that in this model, market efficiency – whereby price tracks any value, at an exponential rate, in the presence of uncertainties (typically of small magnitude) and noise (of arbitrary magnitude) – is equivalent to the traders' aggregate ability to identify asset mispricings and their aggregate behavior incorporating an internal model of the value process dynamics. We have illustrated this result by means of an extensive example. To our knowledge, the price discovery model presented in this article is new, and the proof of the main result appears to be new also in the realm of robust automatic control theory.

The Efficient Market Hypothesis has been criticized for the fact that, in its strongest form, it implies perfectly rational market participants, and that new information should be immediately reflected in asset prices. Our formulation of efficient markets, in contrast to this, specifically avoids any assumption of investor rationality, and we also allow for a gradual value discovery upon news. In particular, we have allowed some investors to be irrational and studied price–value discovery in the presence of such irrational investors and in the presence of model uncertainty. Our main result shows a necessary and sufficient arrangement of traders for market efficiency, assuming that there is no bubble in the asset price. We believe that the significance of this result then lies in the new directions it may provide for studying validity – or the sources of breakdown – of EMH in practice. This can take place not only by studying the conditions of Theorem 1 in practice, but also by applying control-theoretical results in this field. Among several others we mention Maithripala et al. [24] who studied the implications of loss of robust regulation.

The work presented in this paper is mostly theoretical, but it is also possible to study the validity of our modeling approach in practice. Indeed, one can use standard system identification techniques from control theory to determine whether or not any of the typical linear regulator descriptions is adequate for modeling trader behavior in the vicinity of value changes, such as earnings announcements. We expect to report such results in the future. Another important topic for future research is to study how the conditions for market efficiency presented herein may arise as a consequence of agent-based optimization and other essential elements of value formation.

The model presented in this article is, of course, by no means all-inclusive. Although our model allows for parameter uncertainty, it is based on linear dynamics and its structure is fixed in the sense that all the feedback interconnections are explicitly specified. In practice, there may be more elaborate (even time-dependent) interconnections between the blocks of the feedback loop model. The study of such systems is an important topic for future work. Furthermore, the market efficiency concept studied in this article relies on the stabilizability of the closed loop system (i.e. the NBC). In practice, there are additional constraints not considered here, such as finite capital requirement, that restrict the magnitude of the control action $u(t)$. Achieving closed loop stability in such circumstances is a non-trivial task and an important consideration for future work.

Another valid – albeit somewhat philosophical – point of criticism towards the results of this article is the existence and uniqueness of asset value: In practice, asset value can be subjective, and it is usually assumed to be the end result of the auctioning process (as in general equilibrium models). We point out that, while we do assume the existence and uniqueness of value, in our framework individual traders do not have know it for the market to be efficient. Instead, in our model, traders base their trading decisions on value estimates, which may reflect the aforementioned subjective biases. This discussion nonetheless also merits further attention in future research. We conclude by emphasizing that many authors, e.g. Refs. [18,19] have studied particular examples of the general feedback arrangement proposed in this article; we have merely extended those examples into a more general framework which allows for a study of conditions leading to market efficiency.

**Acknowledgments**

The author would like to thank the anonymous reviewers for their constructive criticism and several helpful comments for improving this article.

## Appendix. Proof of the main result

We first prove the necessity of the given conditions for market efficiency.

Assume that $\Lambda S = A_c\Lambda + B_c\Delta$ for some linear maps $\Lambda : \mathcal{W} \to \mathcal{X}$ and $\Delta : \mathcal{W} \to \mathcal{Y}$. Since the market is efficient for arbitrary $P$ and $\tilde{Q}$, the necessary existence of a readability map $K_o : \mathcal{Y} \to \mathbb{R}$ such that $C = K_o\tilde{C}$ and $Q = K_o\tilde{Q}$ now follows immediately from Ref. [25, Theorem 1]. Since $P$ and $\tilde{Q}$ are arbitrary, we can let $\tilde{Q} = \Delta$ and $P = BD_c\tilde{Q} + BC_c\Lambda$. Now let $K$ be any matrix satisfying the conditions (7). Then $Q = K\tilde{Q}$, and it remains to be shown that $K\Delta = 0$.

Define $\mathcal{P} = \begin{pmatrix} BC_c\Lambda & B_c\Delta \end{pmatrix}^T$. Since the spectra of $A_L$ and $S$ are disjoint, there exists a unique matrix $\Pi = \begin{pmatrix} \Pi_1 & \Pi_2 \end{pmatrix}^T : \mathcal{W} \to \mathcal{Z} \times \mathcal{X}$ satisfying the Sylvester equation

$$\Pi S = A_L\Pi + \mathcal{P}. \tag{A.1}$$

A direct calculation shows that the unique solution is given by $\Pi_1 = 0$ and $\Pi_2 = \Lambda$.

Now let $\Theta(t) = \begin{pmatrix} z(t) & x(t) \end{pmatrix}^T$, the state of the closed loop system, which, for the above choices of $P$ and $\tilde{Q}$, is described by the following set of equations:

$$\dot{\Theta}(t) = A_L\Theta(t) + \mathcal{P}w(t), \quad \forall t \geq 0, \qquad \Theta(0) \in \mathcal{Z} \times \mathcal{X} \tag{A.2a}$$

$$\dot{w}(t) = Sw(t), \quad \forall t \geq 0, \qquad w(0) \in \mathcal{W} \tag{A.2b}$$

$$e(t) = \begin{pmatrix} C & 0 \end{pmatrix}\Theta(t) - Qw(t) \quad \forall t \geq 0. \tag{A.2c}$$

Since Condition 1 of Definition 5 is assumed to hold for all $z(0) \in \mathcal{Z}$, $x(0) \in \mathcal{X}$ and $w(0) \in \mathcal{W}$, we can in particular choose $z(0) = 0$ and $x(0) = \Lambda w(0)$. Then it is easy to see (see e.g. Ref. [11, Theorem 6.20]) that

$$\Theta(t) = e^{A_L t}\begin{pmatrix} z(0) \\ x(0) \end{pmatrix} - e^{A_L t}\Pi w(0) + \Pi e^{St}w(0) \tag{A.3}$$

$$= e^{A_L t}\begin{pmatrix} 0 \\ \Lambda w(0) \end{pmatrix} - e^{A_L t}\begin{pmatrix} 0 \\ \Lambda w(0) \end{pmatrix} + \Pi e^{St}w(0) \tag{A.4}$$

$$= \begin{pmatrix} 0 \\ \Lambda \end{pmatrix} e^{St}w(0) \tag{A.5}$$

so that for these initial conditions

$$|e(t)| = |Qw(t)| = |K\Delta e^{St}w(0)| \leq Me^{-at} \quad \forall t \geq 0. \tag{A.6}$$

By the assumptions for $S$ this can only be true if $K\Delta = 0$. This completes the proof of necessity of conditions.

As for sufficiency of the given conditions for market efficiency, let $P$ be arbitrary and let $\tilde{Q}$ be arbitrary such that (7) holds for some readability map $K$ (it exists by assumption). Define $\mathscr{P} = \begin{pmatrix} P - BD_c\tilde{Q} & B_c\tilde{Q} \end{pmatrix}^T$. We will show that $e(t) = Cz(t) - Qw(t)$ decays exponentially independently of the initial conditions in the presence of parameter uncertainties.

Introduce additive perturbations to the matrices in $\Omega$ such that $A^p = A + \Delta_A, B^p = B + \Delta_B, C^p = C + \Delta_C, \tilde{C}^p = \tilde{C} + \Delta_{\tilde{C}}\ C_c^p = C_c + \Delta_{C_c}, D_c^p = D_c + \Delta_{D_c}$, where by assumption $C^p = K\tilde{C}^p$, and all perturbations are small enough to retain closed loop stability. Then, again, there exists a unique matrix $\Pi = \begin{pmatrix} \Pi_1 & \Pi_2 \end{pmatrix}^T : \mathscr{W} \to \mathscr{Z} \times \mathscr{X}$ satisfying the Sylvester equation (A.1) for the perturbed loop matrix $A_L^p$. Working this out yields

$$\Pi_1 S = (A^p + B^p D_c^p \tilde{C}^p)\Pi_1 + B^p C_c^p \Pi_2 + P - B^p D_c^p \tilde{Q} \tag{A.7}$$

$$\Pi_2 S = A_c \Pi_2 + B_c(\tilde{C}^p \Pi_1 - \tilde{Q}) \tag{A.8}$$

so that $0 = K(\tilde{C}^p \Pi_1 - \tilde{Q}) = C^p \Pi_1 - Q$ by the assumption of an internal model.

Similarly as in the above, for any $z(0) \in \mathscr{Z}$, $x(0) \in \mathscr{X}$ and $w(0) \in \mathscr{W}$ we have:

$$\begin{pmatrix} z(t) \\ x(t) \end{pmatrix} = e^{A_L^p t} \begin{pmatrix} z(0) \\ x(0) \end{pmatrix} - e^{A_L^p t} \Pi w(0) + \Pi e^{St} w(0) \tag{A.9}$$

so that

$$e(t) = \begin{pmatrix} C^p & 0 \end{pmatrix} e^{A_L^p t} \begin{pmatrix} z(0) \\ x(0) \end{pmatrix} - \begin{pmatrix} C^p & 0 \end{pmatrix} e^{A_L^p t} \Pi w(0) + \begin{pmatrix} C^p & 0 \end{pmatrix} \Pi e^{St} w(0) - Q e^{St} w(0) \tag{A.10}$$

$$= f(t) + (C^p \Pi_1 - Q) e^{St} w(0) = f(t) + K(\tilde{C}^p \Pi_1 - \tilde{Q}) e^{St} w(0) = f(t) \tag{A.11}$$

for some function $f(\cdot)$, which decays exponentially by the stability of $A_L^p$. This completes the proof.

## References


[1] A. Smith, The Wealth of Nations, Thrifty Books, 2009.
[2] D.A. Walker, Walras's theories of tatonnement, J. Polit. Econ. 95 (4) (1987) 758–774.
[3] P.A. Samuelson, Proof that properly anticipated prices fluctuate randomly, Manag. Rev. 6 (2) (1965) 41–49.
[4] E.F. Fama, Efficient capital markets: A review of theory and empirical work, J. Finance 25 (2) (1970) 383–417.
[5] A.W. Lo, The adaptive markets hypothesis: Market efficiency from an evolutionary perspective, J. Portf. Manag. 30 (5) (2004) 15–29.
[6] A. Lo, A.C. MacKinlay, Stock market prices do not follow random walks: Evidence from a simple specification test, Rev. Financ. Stud. 1 (1) (1988) 41–66.
[7] D. Abreu, M.K. Brunnermeier, Bubbles and crashes, Econometrica 71 (1) (2003) 173–204.
[8] S. Schulmeister, Profitability of technical stock trading: Has it moved to intraday data, Rev. Financ. Econ. 18 (4) (2009) 2720–2727.
[9] J.-P. Bouchaud, J.D. Farmer, F. Lillo, How markets slowly digest changes in supply and demand, in: T. Hens, K. Schenk-Hoppe (Eds.), Handbook of Financial Markets: Dynamics and Evolution, North Holland, 2009, pp. 57–160.
[10] B.A. Francis, W.M. Wonham, The internal model principle of linear control theory, Automatica 12 (1976) 457–465.
[11] E. Immonen, State space output regulation theory for infinite-dimensional linear systems and bounded uniformly continuous exogenous signals (Ph.D. thesis), Tampere University of Technology, 2006, p. 233.
[12] R. Mosetti, On a possible interpretation of market dynamics based on the internal model principle, Appl. Math. Comput. 37 (1990) 1–9.
[13] C.I. Byrnes, A. Isidori, Output regulation for nonlinear systems: an overview, Internat. J. Robust Nonlinear Control 10 (2000).
[14] C.I. Byrnes, A. Isidori, Limit sets, zero dynamics, and internal models in the problem of nonlinear output regulation, IEEE Trans. Automat. Control 48 (10) (2003).
[15] J.-P. Bouchaud, J.D. Farmer, F. Lillo, How markets slowly digest changes in supply and demand, in: T. Hens, K.R. Schenk-Hoppe (Eds.), Handbook of Financial Markets: Dynamics and Evolution, Elsevier, 2009.
[16] P. Bossaerts, The Paradox of Asset Pricing, Princeton University Press, 2002.
[17] A. Gavious, H. Kedar-Levy, The speed of stock price discovery, J. Financ. Intermed. 22 (2) (2013).
[18] J. Alvarez-Ramirez, R. Suarez, C. Ibarra-Valdez, Trading strategies, feedback control and market dynamics, Physica A 324 (2003) 220–226.
[19] J.D. Farmer, S. Joshi, The price dynamics of common trading strategies, J. Econ. Behav. Organ. 49 (2003) 149–171.
[20] C.H. Hommes, Heterogeneous agent models in economics and finance, in: L. Tesfatsion, K.L. Judd (Eds.), Handbook of Computational Economics, Vol. 2, Elsevier, 2006, pp. 1109–1186.
[21] C. Chiarella, X.-Z. Hea, R.C. Zwinkels, Heterogeneous expectations in asset pricing: Empirical evidence from the s&p 500, J. Econ. Behav. Organ. 105 (2014) 1–16.
[22] S.P. Boyd, C.H. Barratt, Linear Controller Design: Limits of Performance, Prentice-Hall, 1991.
[23] E. Immonen, On the internal model structure for infinite-dimensional systems: Two common controller types and repetitive control, SIAM J. Control Optim. 45 (2007) 2065–2093.
[24] D.H.S. Maithripala, J.M. Berg, W.P. Dayawansa, Loss of structurally stable regulation implies loss of stability, IEEE Trans. Automat. Control 48 (2003) 483–487.
[25] B.A. Francis, W.M. Wonham, The internal model principle for linear multivariable regulators, J. Appl. Math. 2 (1975) 170–194.