Intensity-based Credit Risk Models

Reduced-form credit risk models have become standard tools for pricing credit derivatives and for providing a link between credit spreads and default probabilities. In structural models, following the Merton approach [1, 12], default is defined by a firm value hitting a certain barrier. In such an approach, the concept of credit spread is rather abstract since it is not modeled explicitly and therefore is not directly accessible and may also have dynamics that are not completely pleasing. Reduced-form models, however, concentrate on modeling the hazard rate or intensity of default, which is directly linked to the credit spread process. In contrast to a structural approach, the event of default in a reduced-form model comes about as a sudden unanticipated event (although the likelihood of this event may have been changing).

Deterministic Hazard Rates

Risk-neutral Default Probability

The basic idea around pricing default sensitive products is that of considering a risky zero-coupon bond of unit notional and maturity T. We write the payoff at maturity as

$$C(T,T) = \begin{cases} 1 & \text{default} \\ \delta & \text{no default} \end{cases} \tag{1}$$

where δ is an assumed recovery fraction paid immediately in the event of default. The price of a risky cash flow due at time T is then

$$C(t,T) = [S(t,T) + [1 - S(t,T)]\delta]B(t,T) \tag{2}$$

with $B(t,T)$ denoting the risk-free discount factor for time T as seen from time t, $S(t,T)$ is the risk-neutral survival (no default) probability (*see* **Hazard Rate**) in the interval $[t,T]$ or, equivalently, $1 - S(t,T)$ is the risk-neutral default probability. This style of approach was developed by Jarrow and Turnbull [8, 9].

Pricing a Credit Default Swap (CDS)

A credit default swap (CDS) (*see* **Credit Default Swaps**) has become a benchmark product for trading credit risk and hence we base most of our analysis around CDS pricing. Standard assumptions used in pricing CDS include deterministic default probabilities, interest rates, and recovery values (or at least independence between these three quantities). In a CDS contract, the protection buyer will typically pay a fixed periodic premium, X_{CDS}, to the protection seller until the maturity date or the default (credit event) time (T). The present value of these premiums at time t can be written as

$$V_{\mathrm{premium}}(t,T) = \sum_{i=1}^{m} S(t,t_i)B(t,t_i)\Delta_{i-1,i}X_{\mathrm{CDS}} \tag{3}$$

where m is the number of premium payments and $\Delta_{i-1,i}$ represents the day count fraction.

The protection seller in a CDS contract will undertake in the event of a default to compensate the buyer for the loss of notional less some recovery value, δ. The value of the default component obtained by integrating over all possible default times is given by

$$V_{\mathrm{default}}(t,T) = (1-\delta)\int_t^T B(t,u)\,\mathrm{d}S(t,u) \tag{4}$$

Note that due to the required negative slope of $S(t,u)$, this term will be negative; hence, the sum of equations (3) and (4) defines the value of a CDS from a protection provider's point of view.

Defining the Hazard Rate

In pricing a CDS, the main issue is to define $S(t,u)$ for all relevant times in the future, $t \le u \le T$. If we consider default to be a Poisson process driven by a constant intensity of default, then the survival probability is

$$S(t,u) = \exp[-h(u-t)] \tag{5}$$

where h is the intensity of default, often described as the hazard rate. We can interpret h as a forward instantaneous default probability; the probability of default in a small period $\mathrm{d}t$ conditional on no prior default is $h\,\mathrm{d}t$. Default is a sudden unanticipated event (although it may, of course, have been partly anticipated due to a high value of h).

Link from Hazard Rate to Credit Spread

If we assume that CDS premiums are paid continuously,[a] then the value of the premium payments can be written as

$$V_{\text{premium}}(t,T) \approx X_{\text{CDS}} \int_t^T B(t,u)S(t,u)\,\mathrm{d}u \qquad (6)$$

Under the assumption of a constant hazard rate of default, we can write $\mathrm{d}S(t,u) = -hS(t,u)\,\mathrm{d}u$ and the default payment leg becomes

$$V_{\text{default}}(t,T) = -(1-\delta)h \int_t^T B(t,u)S(t,u)\,\mathrm{d}u \qquad (7)$$

The CDS spread will be such that the total value of these components is zero. Hence from $V_{\text{premium}}(t,T) + V_{\text{default}}(t,T) = 0$ we have the simple relationship

$$h \approx \frac{X_{\text{CDS}}}{(1-\delta)} \qquad (8)$$

The above close relationship between the hazard rate and CDS premium (credit spread) is important in that the underlying variable in our model is directly linked to credit spreads observed in the market. This is a key advantage over structural models whose underlying variables are rather abstract and hard to observe.

Simple Formulas

Suppose we define the risk-free discount factors *via* a constant continuously compounded interest rate $B(t,u) = \exp[-r(u-t)]$. We then have closed-form expressions for quantities such as

$$V_{\text{premium}}(t,T)/X_{\text{CDS}} \approx \int_t^T \exp[-(r+h)(u-t)]\,\mathrm{d}u = \frac{1-\exp[-(r+h)(T-t)]}{r+h} \qquad (9)$$

The above expression and equation (8) allow a quick calculation for the value of a CDS, or equivalently a risky annuity or DV01 for a particular credit.

Incorporating Term Structure

For a nonconstant intensity of default, the survival probability is given by

$$S(t,u) = \exp\left[-\int_t^u h(x)\,\mathrm{d}x\right] \qquad (10)$$

To allow for a term structure of credit (e.g., CDS premia at different maturities) and indeed a term structure of interest rates, we must choose some functional form for h. Such an approach is the credit equivalent of yield curve stripping, although due to the illiquidity of credit spreads much less refined, and was first suggested by Li [10]. The single-name CDS market is mainly based around 5-year instruments and other maturities will be rather illiquid. A standard approach is to choose a piecewise constant representation of the hazard rate to coincide with the maturity dates of the individual CDS quotes.

Extensions

Bonds and Basis Issues

Within a reduced-form framework, bonds can be priced in a similar way to CDS:

$$V_{\text{bond}}(t,T) = \sum_{i=1}^{m} S(t,t_i)B(t,t_i)\Delta_{i-1,i}X_{\text{bond}} + S(t,T)B(t,T) - \delta \int_t^T B(t,u)\,\mathrm{d}S(t,u) \qquad (11)$$

The first term above is similar to the default payment on a CDS but the assumption here is that the bond will be worth a fraction δ in default. The second and third terms represent the coupon and principal payments on the bond, respectively. It is therefore possible to price bonds *via* the CDS market (or *vice versa*) and indeed to calibrate a credit curve *via* bonds of different maturities from the same issuer. However, the treatment of bonds and CDS within the same modeling framework must be done with caution. Components such as funding, the CDS delivery option, delivery squeezes, and counterparty risk mean that CDS and bonds of the same issuer will trade with a basis representing nonequal risk-neutral default probabilities. In the context of the formulas, the components creating such a basis would represent different recovery values as well as discount factors when pricing CDS and bonds of the same issuer.

Stochastic Default Intensity

The deterministic reduced-form approach can be extended to accommodate stochastic hazard rates and leads to the following expression for survival probabilities:-

$$S(t,u) = E^{Q}\left[\exp\left[-\int_{t}^{u} h(x)\,\mathrm{d}x\right]\right] \qquad (12)$$

This has led to various specifications for modeling a hazard rate process with parallels with interest-rate models for modeling products sensitive to credit spread volatility with examples to be found in [4, 5, 11]. Jarrow *et al.* [7] (*see* **Jarrow–Lando–Turnbull Model**) have extended such an approach to have a Markovian structure to model credit migration or discrete changes in credit quality that would lead to jump in the credit spread. Furthermore, credit hybrid models with hazard rates correlated to other market variables, such as interest rates, have been introduced. For example, see [13].

Portfolio Approaches

The first attempts at modeling portfolio credit products, such as basket default swaps and CDOs, involved multidimensional hazard rate models. However, it was soon realized that introducing the level of default correlation required to price such products realistically was far from trivial. This point is easily understood by considering that two perfectly correlated hazard rates will not produce perfectly correlated default events and more complex dynamics are required such as those considered by Duffie [3]. Most portfolio credit models have instead followed structural approaches (commonly referred to as *copula models* with the so-called Gaussian copula model becoming the market standard for pricing CDOs; *see* **Gaussian Copula Model**) for reasons of simplicity. Schonbucher and Schubert [14] have shown how to combine intensity and copula models. More recently, the search for more sophisticated portfolio credit risk modeling approaches is largely based around reduced-form models as in [2] and [6] (*see* **Multiname Reduced Form Models**).

Conclusions

We have outlined the specification and usage of reduced-form models for modeling a default process and described the link between the underlying in such a model and market observed credit spreads. We have described the application of such models to vanilla credit derivative structures such as CDS and also more sophisticated structures such as credit spread options, credit hybrid instrument, and portfolio credit products.

End Notes

[a] CDS premiums are typically paid quarterly in arrears but an accrued premium is paid in the event of default to compensate the protection seller for the period for which a premium has been paid. Hence the continuous premium assumption is only a mild approximation.

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Related Articles

Credit Default Swaps; **Duffie–Singleton Model**; **Hazard Rate**; **Multiname Reduced Form Models**; **Nested Simulation**; **Reduced Form Credit Risk Models**.

JON GREGORY

Default Barrier Models

The modeling of default from an economic point of view is a great challenge due to the binary and low probability nature of such an event. Default barrier models provide an elegant solution to this challenge since they link the default event to the point at which some continuously evolving quantity hits a known barrier. In structural models of credit risk (*see* **Structural Default Risk Models**) the process and the barrier are interpreted in terms of capital structure of the firm as the value of the firm and its liabilities. More generally, one can view the process and the barrier as state variables that need not necessarily observable.

Single-name Models

In the classic Merton framework [12], the value of a firm (asset value) is considered to be stochastic and default is modeled as the point where the firm is unable to pay its outstanding liabilities when they mature. The asset value is modeled as a geometric Brownian motion:

$$\frac{\mathrm{d}V_t}{V_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W \tag{1}$$

where μ and σ represent the drift and volatility of the asset, respectively, and $\mathrm{d}W$ is a standard Brownian motion. The original Merton model assumes that a firm has issued only a zero-coupon bond and will not therefore default prior to the maturity of this debt as illustrated in Figure 1. Denoting the maturity and face value of the debt by T and D respectively, the default condition can then be written as $V_T < D$. Through option pricing arguments, Merton then provides a link between corporate debt and equity *via* pricing formulae based on the value of the firm and its volatility (analogously to options being valued from spot prices and volatility). The problem of modeling default is transformed into that of assessing the future distribution of firm value and the barrier where default would occur. Such quantities can be estimated nontrivially from equity data and capital structure information. This is then the key contribution of Merton approach in that low-frequency binary events can be modeling *via* a continuous process and calibrated using high-frequency data.

Practical Extensions of the Merton Approach

The classic Merton approach has been extended by many authors such as Black and Cox [2] and Leland [10]. Commercially, it has been developed by KMV™ (now Moody's KMV) with the aim of predicting default *via* the assessment of 1-year default probability defined as EDF™ (expected default frequency). A more recent and related, although simpler, approach is CreditGrades™.

Moody's KMV Approach. This approach [8, 9] was inspired by the Merton approach to default modeling and aimed to lift many of the stylized assumptions and model the evolution and future default of a company in a realistic fashion. A key aspect of this is to account for the fact that a firm may default at any time but will not necessarily default immediately when they are bankruptcy insolvent (when $V_t < D$). Hence a challenge is to work out exactly where the default barrier is. KMV do this *via* considering both the short-term and long-term liabilities of the firm. Their approach can be broadly summarized in three stages:

- estimation of the market value and volatility of a firm's assets;
- calculation of the distance to default which is an index measure of default risk; and
- scaling of the distance to default to the actual probability of default using a default database.

The distance to default (DD) measure, representing a standardized distance from which a firm is above its default threshold, is defined by[a]

$$DD = \frac{\ln(V/D) + (\mu - 0.5\sigma^2)T}{\sigma\sqrt{T}} \tag{2}$$

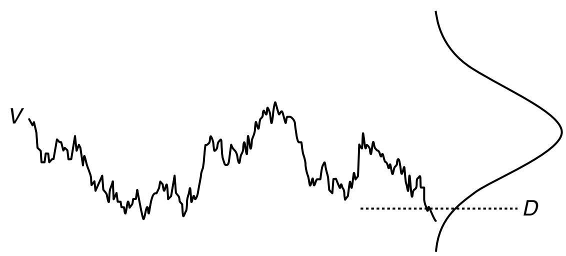

Figure 1 Illustration of the traditional Merton approach to modeling default based on the value of the firm being below the face value of debt at maturity

The default probability would then be given by $p_d = \Phi(-DD)$. A key element of the approach is to recognize the model risk inherent in this approach and rather to estimate the default probability empirically from many years of default history (and the calculated DD variables). We therefore ask ourselves the following question: for a firm with a DD of 4.0 (say), how often have firms with the same DD defaulted historically? The answer is likely to be considerably higher than the theoretical result of $\Phi(-4.0) = 0.003\%$. This mapping of DD to actual default probability could be thought of as an empirical correction for the non-Gaussian behavior of firm value.

CreditGrades Approach. The aim of CreditGrades is rather similar to that of KMV except that the modeling framework [3] is rather simpler, in particular without using empirical data in order to map to an eventual default probability. In the Credit Grades approach, the default barrier is given by

$$LD = \overline{L} D e^{\lambda Z - \lambda^2/2} \quad (3)$$

where Z is a standard normal variable, D is the "debt per share", $\overline{L}$ is an average recovery level, and λ creates an uncertainty in the default barrier. The level of the default barrier and the asset return are independent. Hence the main differences between the traditional Merton approach and CreditGrades is that the latter approach assumes that default can occur at any time when the asset process has dropped to a level of LD, whereas the Merton framework assumes $\overline{L} = 1$ and $\lambda = 0$ and no default prior to the maturity of the debt. CreditGrades recommends values of $\overline{L} = 0.5$ and $\lambda = 0.3$. A sensitivity analysis of these parameters should give the user a very clear understanding of the uncertainties inherent in estimating default probability.

Portfolio Models

While default barrier models have proved very useful for assessing single-name default probability and supporting trading strategies such as capital structure arbitrage, arguably an even more significant development has been their application in credit portfolio models. The basic strength of the default barrier approach is to provide the transformation necessary to model default events *via* a multivariate normal distribution driven by asset correlations. The intuition of the approach makes it possible to add complexities such as credit migrations and stochastic recovery rates into the model.

Default Correlation

Consider modeling the joint default probability of two entities. Using the standard definition of a correlation coefficient, we can write the joint default probability as

$$p_{AB} = p_A p_B + \rho_{AB}\sqrt{p_A(1-p_A)p_B(1-p_B)} \quad (4)$$

where p_A and p_B are the individual default probabilities and ρ_{AB} is the default correlation. Assuming, without loss of generality, that $p_A \leq p_B$ and since the joint default probability can be no greater than the smaller of the individual default probabilities, we have

$$\rho_{AB} = \frac{p_{AB} - p_A p_B}{\sqrt{p_A(1-p_A)p_B(1-p_B)}} \leq \frac{p_A - p_A p_B}{\sqrt{p_A(1-p_A)p_B(1-p_B)}} = \sqrt{\frac{p_A(1-p_B)}{p_B(1-p_A)}} \quad (5)$$

This shows that the default correlation cannot be +1 (or indeed *via* a similar argument −1) unless the individual default probabilities are equal. There is therefore a maximum (and minimum) possible default correlation that changes with the underlying default probabilities. This suggests a need for more economic structure to model joint default probability.

Default Barrier Approach

Suppose that we write default as being driven by a standard Gaussian variable X_i being below a certain level $k = \Phi^{-1}(p)$. We can interpret X as being an asset return in the classic Merton sense, with k being a default barrier. Now joint default probability is readily defined *via* a bivariate Gaussian distribution:

$$p_{AB} = \Phi_2\left(\Phi^{-1}(p_A), \Phi^{-1}(p_B); \lambda_{AB}\right) \quad (6)$$

where Φ_2 is a cumulative bivariate cumulative distribution function and λ_{AB} is the "asset correlation".

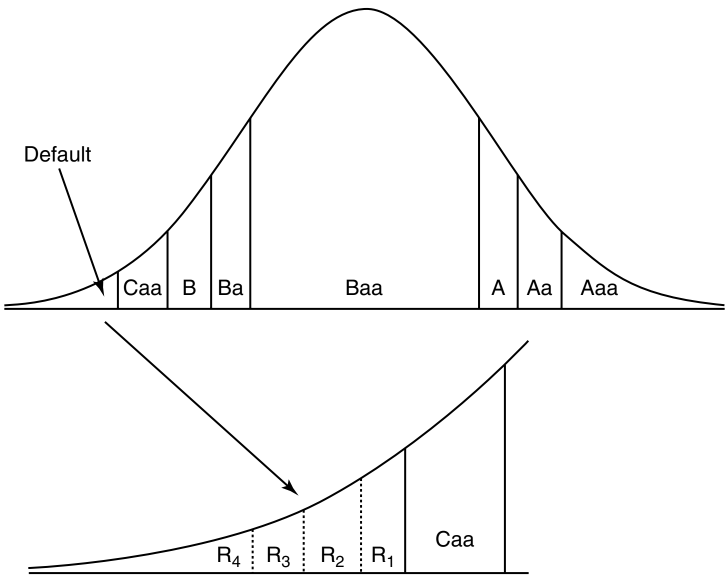

Figure 2 Illustration of the mapping of default and credit migrations thresholds as used in the CreditMetrics approach. The default region is also shown with additional thresholds corresponding to different recovery values with $R_1 < R_2 < R_3 < R_4$

Multiple names can be handled *via* a multivariate Gaussian distribution[b] with Monte Carlo simulation or various factor-type approaches used for the calculation of multiple defaults and/or losses.

Although there is a clear link between this simple approach and the multidimensional Merton model, we have ignored the full path of the asset value process and linked default to just a single variable X_i. A more rigorous time-dependent approach can be found in [7], which is much more complex and time consuming to implement. In practice, the one-period approach is rather similar to the full approach for relative small default probabilities.

CreditMetrics

CreditMetrics [6], first published in 1997, is a credit portfolio model based on the multivariate normal default barrier approach. This framework assumes a default barrier as described above and also considers the mapping of credit migration probabilities onto the same normal variable. A downgrade can therefore be seen as a less extreme move not causing default. In addition to credit migrations, one can also superimpose different recovery rates onto the same mapping so that there is more than one default barrier with lower barriers representing more severe default and therefore a lower recovery value; for example, see [1]. An illustration of the mapping is shown in Figure 2.

Regulatory Approaches

Basel 2. A key strength of the above framework is that defaults, credit migrations, and recovery rates can be modeled within a single intuitive framework with correlation parameters estimated from equity data. While other credit portfolio modeling frameworks have been proposed, the CreditMetrics style approach has been the most popular. Indeed, the Basel 2 formula [4] can be seen as arising from a simplified version of this approach with the following assumptions:

- no credit migration or stochastic recovery and
- infinitely large homogeneous portfolio.

Rating Agency Approaches to Structured Finance. With the massive growth of the collateralized debt obligation (CDO) came a need for rating agencies (*see* **Structured Finance Rating Methodologies**) to model the risk inherent in a CDO structure with a view to assigning a rating to some or all of the tranches of the CDO capital structure. Rating a tranche of a CDO is essentially the same problem as estimating capital on a credit portfolio and hence it may come as no surprise that the rating agencies models were based on default barrier approaches. The rating agencies models can be thought of as therefore heavily following the CreditMetrics approach. The credit crisis of 2007 brought very swift criticism of rating agency modeling approaches to rating all types of structure finance and CDO structures. This was related largely to poor assessment of the model parameters (specifically rather optimistic default probabilities and correlation assumptions) rather than a failure of the model itself.

CDO Pricing

A final and perhaps most exciting (although not for necessarily positive reasons) application of the default barrier approach is in the pricing of synthetic CDO structures. The market standard approach for pricing CDOs follows the work of Li [11] (*see* **Gaussian Copula Model**) who models time of default in a multivariate normal framework:

$$\Pr(T_A < 1, T_B < 1) = \Phi_2\left(\Phi^{-1}(F_A(T_A)), \Phi^{-1}(F_B(T_B)); \gamma\right) \quad (7)$$

where F_A and F_B are the distribution functions for the survival times T_B and T_B and γ is a correlation parameter. At first glance, although this uses the same multivariate distribution, or copula[c], this approach initially does not seem to be a default barrier model. However, as noted in [11], for a single period, the approaches are identical. Furthermore, as shown by Laurent and Gregory [5], the pricing of a synthetic CDO requires just the knowledge of loss distributions at each date up to the contractual maturity date (and not any further dynamical information). Hence, we can think of the Li approach as being again similar[d] to the traditional framework of credit portfolio modeling, following CreditMetrics and ultimately inspired by the Merton approach to modeling default *via* the hitting time of a barrier. The recent strong criticism linking the model in [11] to the credit crisis [13] does not fairly consider the rather naïve calibration use of the model that has caused many of the problems in structured finance.

Conclusions

We have described the range of default barrier models used in default probability estimation, capital structure trading, credit portfolio management, regulatory capital calculations, and pricing and rating CDO products. The intuition that default can be modeled as a hitting of a barrier has been crucial to the rapid development of credit risk models. For credit portfolio risk in particular, the default barrier approach has been key to the development of models for many different purposes, driven from the same underlying structural framework. Given that some applications of the approach (most notably rating agency models and CDO pricing) have received large criticism, it is worth pointing out that one can only discredit the entire framework (including any multidimensional Merton approach) or realize that it is a misuse of the model rather than the model itself that lies at the heart of the problems.

End Notes

[a.] In the proprietary Moody's KMV implementation, the default point is not the face value of debt but the current book value of their liabilities. This is often computed as short-term liabilities plus half long-term liabilities.

[b.] It should be noted that alternatives to a Gaussian distribution (e.g., student-t) can and have been considered although the Gaussian approach has remained most common.

[c.] This approach has become known as the *Gaussian copula model* which is perhaps confusing since the key point of the approach is the representation of the joint distribution of default times and not the choice of a Gaussian copula or multivariate distribution.

[d.] Li was at the time working at JP Morgan and so this is not surprising.

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Related Articles

Credit Risk; **Structural Default Risk Models**.

JON GREGORY

Multiname Reduced Form Models

Currently, there are three established approaches for describing the default of a single credit: (i) reduced-form; (ii) structural; and (iii) hybrid. It has been an outstanding goal for many researchers to extend these approaches to baskets of several (potentially many) credits. In this article, we concentrate on the reduced-form approach and show how it works in single-name and multiname settings.

Single-name Intensity Models

For a single name, the main assumptions of the reduced-form model are as follows [8, 9, 12]). The name defaults at the first time a Cox process jumps from 0 to 1. The default intensity (hazard rate) $X(t)$ of this process is governed by a mean-reverting nonnegative jump-diffusion process

$$\mathrm{d}X(t) = f(t, X(t))\,\mathrm{d}t + g(t, X(t))\,\mathrm{d}W(t) + J\,\mathrm{d}N(t), \qquad X(0) = X_0 \tag{1}$$

where $W(t)$ is a standard Wiener process, $N(t)$ is a Poisson process with intensity $\lambda(t)$, and J is a positive jump distribution; W, N, J are mutually independent. It is clear that we have to impose the following constraints:

$$f(t, 0) \geq 0, \quad f(t, \infty) < 0, \quad g(t, 0) = 0 \tag{2}$$

plus a number of other technical conditions to ensure that $X(t)$ stays nonnegative and is mean reverting.

For analytical convenience (rather than for stronger reasons), it is customary to assume that X is governed by the square-root stochastic differential equation (SDE):

$$\mathrm{d}X(t) = \kappa(\theta(t) - X(t))\,\mathrm{d}t + \sigma\sqrt{X(t)}\,\mathrm{d}W(t) + J\,\mathrm{d}N(t), \quad X(0) = X_0 \tag{3}$$

with exponential (or hyperexponential) jump distribution [4]. However, for practical purposes it is more convenient to consider discrete jump distributions with jump values $J_m > 0$, $1 \leq m \leq M$, occurring with probabilities $\pi_m > 0$; such distributions are more flexible than parametric ones because they allow one to place jumps where they are needed.

In this framework, the survival probability of the name from time 0 to time T has the form

$$q(0, T) = \mathbb{E}_0\left\{\mathrm{e}^{-\int_0^T X(t')\,\mathrm{d}t'}\right\} = \mathbb{E}_0\left\{\mathrm{e}^{-Y(T)}\right\} \tag{4}$$

where $Y(t)$ is governed by the following degenerate SDE:

$$\mathrm{d}Y(t) = X(t)\,\mathrm{d}t, \quad Y(0) = 0 \tag{5}$$

More generally, the survival probability from time t to time T conditional on no default before time t has the form

$$\begin{aligned} q(t, T \mid X(t), Y(t)) &= \mathbb{1}_{(\tau>t)}\mathbb{E}_t\left\{\mathrm{e}^{-\int_t^T X(t')\,\mathrm{d}t'}\,\middle|\, X(t), Y(t)\right\} \\ &= \mathrm{e}^{Y(t)}\mathbb{1}_{(\tau>t)}\mathbb{E}_t\left\{\mathrm{e}^{-Y(T)}\,\middle|\, X(t), Y(t)\right\} \end{aligned} \tag{6}$$

where τ is the default time and $\mathbb{1}_{(\tau>t)}$ is the corresponding indicator function. This expectation, and, more generally, expectations of the form $\mathbb{E}_t\left\{\mathrm{e}^{-\xi Y(T)}\,\middle|\, X(t), Y(t)\right\}$, can be computed by solving the following augmented partial differential equation (PDE) (see [10], Chapter 13):

$$\mathcal{L}V(t, T, X, Y) + XV_Y(t, T, X, Y) = 0 \tag{7}$$

$$V(T, T, X, Y) = \mathrm{e}^{-\xi Y} \tag{8}$$

where

$$\mathcal{L}V \equiv V_t + \kappa(\theta(t) - X)V_X + \frac{1}{2}\sigma^2 XV_{XX} + \lambda\sum_m \pi_m\left[V(X + J_m) - V(X)\right] \tag{9}$$

Specifically, the following relation holds:

$$\mathbb{E}_t \left\{ \mathrm{e}^{-\xi Y(T)} \middle| X(t), Y(t) \right\} = V(t, T, X(t), Y(t)) \tag{10}$$

The corresponding solution can be written in the so-called affine form:

$$V(t, T, X, Y) = \mathrm{e}^{a(t,T,\xi)+b(t,T,\xi)X-\xi Y} \tag{11}$$

where a, b are functions of time governed by the following system of ordinary differential equations (ODEs):

$$\begin{cases} \dfrac{\mathrm{d}a(t,T,\xi)}{\mathrm{d}t} = -\kappa\theta(t)\, b(t,T,\xi) \\ \qquad -\lambda \sum_m \pi_m \left[\mathrm{e}^{J_m b(t,T,\xi)} - 1 \right] \\ \dfrac{\mathrm{d}b(t,T,\xi)}{\mathrm{d}t} = \xi + \kappa b(t,T,\xi) \\ \qquad -\dfrac{1}{2}\sigma^2 b^2(t,T,\xi) \end{cases} \tag{12}$$

$$a(T, T, \xi) = 0, \quad b(T, T, \xi) = 0 \tag{13}$$

While in the presence of discrete jumps this system cannot be solved analytically, it is very easy to solve it numerically *via* the standard Runge–Kutta method. The survival probability $q(0, T)$ and default probability $p(0, T)$ have the form

$$\begin{aligned} q(0,T) &= \mathrm{e}^{a(0,T,1)+b(0,T,1)X_0} \\ p(0,T) &= 1 - q(0,T) = 1 - \mathrm{e}^{a(0,T,1)+b(0,T,1)X_0} \end{aligned} \tag{14}$$

Assuming for simplicity that the short interest rate $r(t)$ is deterministic and the protection payments are made continuously, we can write the value U of a credit default swap (CDS) paying an up-front amount υ and a coupon s in exchange for receiving $1 - R$ (where R is the default recovery) on default as follows:

$$U = -\upsilon + V(0, X_0) \tag{15}$$

Here, $V(t, X)$ solves the following pricing problem:

$$\mathcal{L}V(t, X) - (r + X)V(t, X) = s - (1 - R)X \tag{16}$$

$$V(T, X) = 0 \tag{17}$$

where $\mathcal{L}$ is given by expression (9). Using Duhamel's principle, we obtain the following expression for V:

$$\begin{aligned} V(t, X) = & -s \int_t^T D(t, t')\, \mathrm{e}^{a(t,t',1)+b(t,t',1)X} \,\mathrm{d}t' \\ & - (1-R) \int_t^T D(t, t')\, \mathrm{d}\left[\mathrm{e}^{a(t,t',1)+b(t,t',1)X} \right] \end{aligned} \tag{18}$$

where

$$D(t, t') = \mathrm{e}^{-\int_t^{t'} r(t'')\,\mathrm{d}t''} \tag{19}$$

is the discount factor between two times t and t'. Accordingly,

$$\begin{aligned} U = & -\upsilon - s \int_0^T D(0, t')\left(1 - p(0, t')\right) \mathrm{d}t' \\ & + (1 - R) \int_0^T D(0, t')\, \mathrm{d}p(0, t') \end{aligned} \tag{20}$$

For a given up-front payment υ, we can represent the corresponding par spread $\hat{s}$ (i.e., the spread that makes the value of the corresponding CDS zero) as follows:

$$\hat{s}(T) = \frac{-\upsilon + (1 - R) \int_0^T D(0, t')\, \mathrm{d}p(0, t')}{\int_0^T D(0, t')\left(1 - p(0, t')\right) \mathrm{d}t'} \tag{21}$$

It is clear that the numerator represents the payout in the case of default, while the denominator represents the risky DV_{01}. Conversely, for a given spread we can represent the par up-front payment in the form

$$\hat{\upsilon} = -s(T)\int_0^T D\left(0,t'\right)\left(1-p\left(0,t'\right)\right)\,\mathrm{d}t' + (1-R)\int_0^T D\left(0,t'\right)\,\mathrm{d}p\left(0,t'\right) \tag{22}$$

In these formulas, we implicitly assume that the corresponding CDS is fully collateralized, so that in the event of default $1-R$ is readily available. Shortly, we will evaluate CDS spreads in the presence of the counterparty risk.

In general, there is not enough market information to calibrate the diffusion and jump parts. So, typically, they are viewed as given constants, and the mean-reversion level $\theta(t)$ is calibrated in such a way that the whole par spread curve is matched.

Multiname Intensity Models

The Two-name Case

It is very tempting to extend the above framework to cover several correlated names. For example, consider two credits, A, B and assume for simplicity that their default intensities coincide,

$$X_A(t) = X_B(t) = X(t) \tag{23}$$

and both names have the same recovery $R_A = R_B = R$. For a given maturity T, the default event correlation ρ is defined as follows:

$$\rho(0,T) = \frac{P(\tau_A \le T, \tau_B \le T) - P(\tau_A \le T)\,P(\tau_B \le T)}{\sqrt{P(\tau_A \le T)(1-P(\tau_A \le T))\,P(\tau_B \le T)(1-P(\tau_B \le T))}} \tag{24}$$

$$\rho(0,T) = \frac{p_{AB}(0,T) - p_A(0,T)\,p_B(0,T)}{\sqrt{p_A(0,T)(1-p_A(0,T))\,p_B(0,T)(1-p_B(0,T))}} \tag{25}$$

where τ_A, τ_B are the default times, and

$$p_A(0,T) = P(\tau_A \le T), \quad p_B(0,T) = P(\tau_B \le T)$$
$$p_{AB}(0,T) = P(\tau_A \le T, \tau_B \le T) \tag{26}$$

It is clear that

$$p_A(0,T) = p_B(0,T) = p(0,T) = 1 - \mathrm{e}^{a(0,T,1)+b(0,T,1)X_0} \tag{27}$$

Simple calculation yields

$$p_{\mathrm{AB}}(0,T) = \mathbb{E}_0\left\{\mathrm{e}^{-\int_0^T (X_A(t')+X_B(t'))\,\mathrm{d}t'}\right\} + p_A(0,T) + p_B(0,T) - 1 = \mathbb{E}_0\left\{\mathrm{e}^{-2\int_0^T X(t')\,\mathrm{d}t'}\right\} + 2p(0,T) - 1 \tag{28}$$

so that

$$\rho(0,T) = \frac{\mathbb{E}_0\left\{\mathrm{e}^{-2\int_0^T X(t')\,\mathrm{d}t'}\right\} - (1-p(0,T))^2}{p(0,T)(1-p(0,T))} = \frac{\mathrm{e}^{a(0,T,2)+b(0,T,2)X_0} - \mathrm{e}^{2a(0,T,1)+2b(0,T,1)X_0}}{\left(1-\mathrm{e}^{a(0,T,1)+b(0,T,1)X_0}\right)\mathrm{e}^{a(0,T,1)+b(0,T,1)X_0}} \tag{29}$$

It turns out that in the absence of jumps, the corresponding event correlation is very low [12]. However, if large positive jumps are added (while overall survival probability is preserved), then correlation can increase all the way to one. Assuming that $T = 5y$, $\kappa = 0.5$, $\sigma = 7\%$, and $J = 5.0$, we illustrate this observation in Figure 1.

In the two-name portfolio, we can define two types of CDSs which depend on the correlation: (i) the first-to-default (FTD) swap; (ii) the

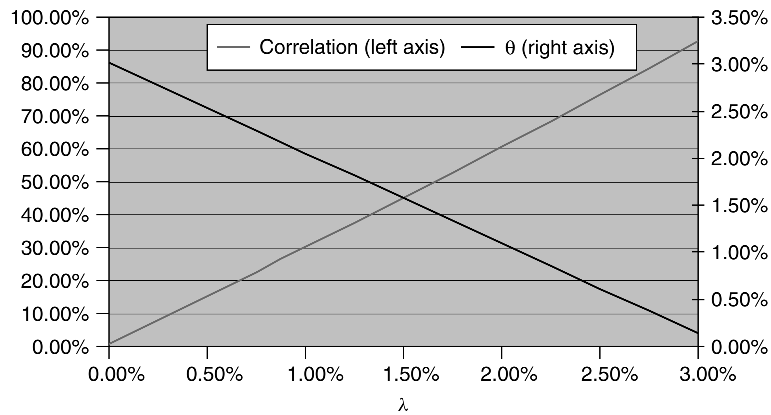

Figure 1 Correlation ρ and mean-reversion level $\theta = X_0$ as functions of jump intensity λ. Other parameters are as follows: $T = 5y$, $\kappa = 0.5$, $\sigma = 7\%$, and $J = 5.0$

second-to-default (STD) swap. The corresponding par spreads (assuming that there are no up-front payments) are

$$\hat{s}_1(T) = \frac{(1-R)\int_0^T D(0,t')\,\mathrm{d}\left[1-\mathrm{e}^{a(0,t',2)+b(0,t',2)X_0}\right]}{\int_0^T D(0,t')\,\mathrm{e}^{a(0,t',2)+b(0,t',2)X_0}\,\mathrm{d}t'} \tag{30}$$

$$\hat{s}_2(T) = \frac{(1-R)\int_0^T D(0,t')\,\mathrm{d}\left[1-\left(2\mathrm{e}^{a(t',1)+b(t',1)X_0}-\mathrm{e}^{a(t',2)+b(t',2)X_0}\right)\right]}{\int_0^T D(0,t')\left(2\mathrm{e}^{a(t',1)+b(t',1)X_0}-\mathrm{e}^{a(t',2)+b(t',2)X_0}\right)\mathrm{d}t'} \tag{31}$$

It is clear that the relative values of $\hat{s}_1, \hat{s}_2$ very strongly depend on whether or not jumps are present in the model (see Figure 2).

However, an even more important application of the above model is the evaluation of counterparty effects on fair CDS spreads. Let us assume that name A has written a CDS on reference name B. It is clear that the pricing problem for the value of the uncollateralized CDS $\tilde{V}$ can be written as follows:

$$\mathcal{L}\tilde{V}(t,X) - (r+2X)\tilde{V}(t,X) = s - (1-R)X - (RV_+(t,X) + V_-(t,X))X \tag{32}$$

where V is the value of a fully collateralized CDS on name B with spread s, and $V_+ = \max\{V, 0\}$, $V_- = \min\{V, 0\}$. It is clear that the discount rate is increased from $r + X$, in equation (16), to $r + 2X$, in equation (32), since there are two cases when the uncollateralized CDS can be terminated due to default: when the reference name B defaults and when the issuer A defaults. The terms on the right represent a continuous stream of coupon payments, the amount received if B defaults before A, and the amount received (or paid) in case when A defaults before B. Although equation (32) is no longer analytically solvable, it can be solved numerically *via*, say, an appropriate modification of the classical Crank–Nicholson method. It turns out that in the presence of jumps the value of the fair par spread goes down dramatically.

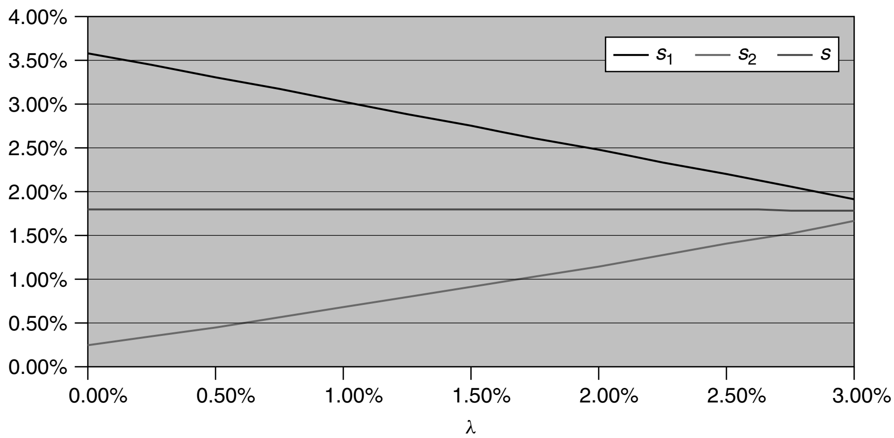

Figure 2 FTD spread $\hat{s}_1$, STD spread $\hat{s}_2$, and single-name CDS spread $\hat{s}$ as functions of jump intensity λ. Other parameters are the same as in Figure 1. It is clear that jumps are necessary to have $\hat{s}_1$ and $\hat{s}_2$ of similar magnitudes

The Multiname Case

The above modeling framework has been expanded in various directions and used as a basis for several coherent intensity-based models for credit baskets; see [2, 3, 6, 7, 11].

To start, we briefly summarize the affine jump-diffusion model of Duffie–Garleanu [3] and Mortensen [11]. Consider a basket of N names with equal unit notionals and equal recoveries R. Let us assume that the corresponding default intensities can be decomposed as follows:

$$X_i(t) = \beta_i X_c(t) + \tilde{X}_i(t) \quad (33)$$

where X_c is the common intensity driven by the following SDE:

$$\begin{aligned} \mathrm{d}X_c(t) &= \kappa_c(\theta_c - X_c(t))\,\mathrm{d}t + \sigma_c\sqrt{X_c(t)}\,\mathrm{d}W_c(t) \\ &\quad + J_c\,\mathrm{d}N_c(t) \\ X_c(0) &= X_{c0} \end{aligned} \quad (34)$$

while $\tilde{X}_i$ are idiosyncratic intensities driven by similar SDEs:

$$\begin{aligned} \mathrm{d}\tilde{X}_i(t) &= \kappa_i\left(\theta_i - \tilde{X}_i(t)\right)\mathrm{d}t + \sigma_i\sqrt{\tilde{X}_i(t)}\,\mathrm{d}W_i(t) \\ &\quad + \tilde{J}_i\,\mathrm{d}N_i(t) \\ \tilde{X}_i(0) &= \tilde{X}_{i0} \end{aligned} \quad (35)$$

Here, $1 \le i \le N$. The processes $\tilde{X}_i(t)$, $X_c(t)$ are assumed to be independent. In this formulation, β_i are similar to the β_i appearing in the capital asset pricing model (CAPM). We note that θ_c, θ_i are assumed to be constant. In the original Duffie–Garleanu formulation, it was assumed that all $\beta_i = 1$. However, this assumption is very restrictive since it limits the magnitude of the common factor by the size of the lowest spread X_i, so that, in general, high correlation cannot be achieved. It was lifted in the subsequent paper by Mortensen. Of course, to preserve analyticity, one needs to impose very rigid conditions on the coefficients of the corresponding SDEs, since, in general, the sum of two affine processes is not an affine process. Specifically, the following should hold:

$$\kappa_i = \kappa_c = \kappa, \quad \sigma_i = \sqrt{\beta_i}\sigma_c, \quad \lambda_i = \lambda, \quad J_{im} = \beta_i J_{cm} \quad (36)$$

Even when the above constraints are satisfied, there are too many free parameters in the model. A reduction in their number is achieved by imposing the following constraints:

$$\frac{\beta_i\theta_c}{\beta_i\theta_c + \theta_i} = \frac{\lambda_c}{\lambda_c + \lambda} = \frac{X_c(0)}{X_c(0) + X_{\text{ave}}(0)} = \omega \quad (37)$$

where ω is a correlation-like parameter representing the systematic share of intensities, and $X_{\text{ave}}(0)$ is the average of $X_i(0)$. When ω is low, the dynamics of intensities is predominantly idiosyncratic, and it is systemic when ω is close to one.

Provided that equation (36) is true, the affine ansatz still holds, so that survival probabilities of individual names can be written in the form

$$\begin{aligned} q_i\left(t, T \mid X_i(t)\right) &= \mathbb{I}_{(\tau_i>t)} \mathbb{E}_t\left\{\left. \mathrm{e}^{-\int_t^T X_i(t')\,\mathrm{d}t'} \right| X_i(t)\right\} \\ &= \mathbb{I}_{(\tau_i>t)} \mathbb{E}_t\left\{\left. \mathrm{e}^{-\beta_i[Y_c(T)-Y_c(t)]} \right| X_c(t)\right\} \\ &\quad \times \mathbb{E}_t\left\{\left. \mathrm{e}^{-\left[\tilde{Y}_i(T)-\tilde{Y}_i(t)\right]} \right| \tilde{X}_i(t)\right\} \\ &= \mathbb{I}_{(\tau_i>t)} \mathrm{e}^{a_c(t,T,\beta_i)+b_c(t,T,\beta_i)X_c(t)+a_i(t,T,1)+b_i(t,T,1)\tilde{X}_i(t)} \end{aligned} \quad (38)$$

Moreover, conditioning the dynamics of spreads on the common factor $Y_c(T)$,we can write idiosyncratic survival probabilities as follows:

$$q_i\left(t, T \mid \tilde{X}_i(t), Y_c(T)\right) = \mathbb{I}_{(\tau_i>t)} \mathrm{e}^{-\beta_i[Y_c(T)-Y_c(t)]+a_i(t,T,1)+b_i(t,T,1)\tilde{X}_i(t)} \quad (39)$$

$$q_i\left(0, T \mid \tilde{X}_{i0}, Y_c(T)\right) = \mathrm{e}^{-\beta_i Y_c(T)+a_i(0,T,1)+b_i(0,T,1)\tilde{X}_{i0}} \quad (40)$$

First, we perform the calibration of the model parameters to fit $1y$ and $5y$ CDS spreads for individual names. Once this calibration is performed, we can apply the usual recursion and calculate the conditional probability of loss of exactly n names, $0 \le n \le N$, in the corresponding portfolio, or, equivalently, of the loss of size $(1-R)\,n$, which we denote as $p(0, T, n \mid Y)$.

For a tranche of the portfolio which covers losses from the attachment point α to the detachment point δ, $0 \le \alpha < \delta \le 1$, the relative tranche loss is defined as follows:

$$\Lambda_{\alpha,\delta}(L) = \frac{\max\{\min\{L, \delta N\} - \alpha N, 0\}}{(\delta-\alpha)\,N} \quad (41)$$

Its conditional expectation has the form

$$p_{\alpha,\delta}(0, T \mid Y) = \sum_{n=0}^{N} \Lambda_{\alpha,\delta}\left((1-R)\,n\right) p(0, T, n \mid Y) \quad (42)$$

In order to find the unconditional expectation, we have to integrate $p_{\alpha,\delta}(0, T \mid Y)$ with respect to the distribution $f(Y)$ of the common factor Y. The latter distribution can be found *via* the inverse Laplace transform of the function

$$\phi(\xi) = \int_0^\infty \mathrm{e}^{-\xi Y} f(Y)\,\mathrm{d}Y = \mathrm{e}^{a_c(0,T,\xi)+b_c(0,T,\xi)X_{c0}} \quad (43)$$

by numerically calculating the Bromwich integral in the complex plane

$$\begin{aligned} f(Y) &= \frac{1}{2\pi\mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{\xi Y} \phi(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{\xi Y + a_c(0,T,\xi)+b_c(0,T,\xi)X_{c0}}\,\mathrm{d}\xi \end{aligned} \quad (44)$$

Both standard and more recent methods allow one to calculate the inverse transform without too much difficulty; see, for example, [1]. Finally, we calculate the unconditional expectation of the tranche loss by performing integration over the common factor:

$$p_{\alpha,\delta}(0, T) = \int_0^\infty p_{\alpha,\delta}(0, T \mid Y) f(Y)\,\mathrm{d}Y \quad (45)$$

Knowing this expectation, we can represent par spread and par up-front for the tranche in question by slightly generalizing formulas (21) and (22). In other words,

$$s_{\alpha,\delta}(T) = \frac{-\upsilon + \int_0^T D\left(0, t'\right) \mathrm{d}p_{\alpha,\delta}\left(0, t'\right)}{\int_0^T D\left(0, t'\right)\left(1 - p_{\alpha,\delta}\left(0, t'\right)\right) \mathrm{d}t'} \quad (46)$$

$$\begin{aligned} \upsilon = & -s_{\alpha,\delta}(T) \int_0^T D\left(0, t'\right)\left(1 - p_{\alpha,\delta}\left(0, t'\right)\right) \mathrm{d}t' \\ & + \int_0^T D\left(0, t'\right) \mathrm{d}p_{\alpha,\delta}\left(0, t'\right) \end{aligned} \quad (47)$$

Equity tranches with $\alpha = 0$, $\delta < 1$ (and, in some cases, other junior tranches) are traded with a fixed spread, say $s = 5\%$, and an up-front determined by formula (47); more senior tranches are traded with zero up-front and spread determined by formula (46).

Treatment of super-senior tranches with $\delta = 1$ has to be slightly modified, but we do not discuss the corresponding details for the sake of brevity.

The affine jump-diffusion model allows one to price tranches of standard on-the-run indices, such as CDX and iTraxx with reasonable (but not spectacular) accuracy, and can be further used to price bespoke tranches; however, one can argue that the presence of the stochastic idiosyncratic components makes it unnecessarily complex. In any case, the very rigid relationships between the model parameters suggest that the choice of these components is fairly limited and rather artificial.

Two models without stochastic idiosyncratic components were independently proposed in the literature. The first one, due to Chapovsky *et al.* [2], assumes purely deterministic idiosyncratic components, and represents q_i as follows:

$$q_i\left(0, T \mid Y_c(T)\right) = \mathrm{e}^{-\beta_i(T)Y_c(T)+\xi_i(T)} \tag{48}$$

where, X_c, Y_c are driven by SDEs (1) and (5), while $\xi_i(T)$ is calibrated to the survival probabilities of individual names. The second one, due to Inglis–Lipton [6], models conditional survival probabilities directly, and postulates that $q_i\left(0, T \mid Y_c\right)$ can be represented in the logit form

$$q_i\left(0, T \mid Y_c(T)\right) = \mathbb{E}_t\left\{\frac{1}{1+\mathrm{e}^{Y_c(T)+\chi_i(T)}}\right\} \tag{49}$$

We now describe the Inglis–Lipton model in some detail. To calibrate the model to individual CDS spreads, we need to solve the following pricing problem:

$$\hat{\mathcal{L}}V(t, X, Y) + XV_Y(t, X, Y) = 0 \tag{50}$$

$$V(T, X, Y) = \frac{1}{1+e^Y} \tag{51}$$

where

$$\hat{\mathcal{L}}V \equiv V_t + f(t, X)V_X + \frac{1}{2}g^2(t, X)V_{XX} + \lambda\sum_m \pi_m\left[V(X+J_m) - V(X)\right] \tag{52}$$

and determine $\chi_i(T)$ from the following algebraic equation (rather than a PDE):

$$V\left(0, 0, \chi_i(T)\right) = q_i(0, T), \quad 1 \le i \le N \tag{53}$$

As before, we can easily calculate the probability of loss of exactly n names, $0 \le n \le N$, $p\left(0, T, n \mid Y\right)$, conditional on Y. We can then solve the pricing equation (50) with the terminal condition

$$V_{\alpha,\delta}(T, X, Y) = p_{\alpha,\delta}\left(0, T \mid Y\right) \tag{54}$$

and find the expected losses for an individual tranche at time 0:

$$p_{\alpha,\delta}(0, T) = V_{\alpha,\delta}(0, X_0, 0) \tag{55}$$

Here, $p_{\alpha,\delta}\left(0, T \mid Y\right)$, $p_{\alpha,\delta}(0, T)$ have the same meaning as in equations (42) and (45). In order to price senior tranches rare but large jumps are necessary. Since, as a rule, we need to analyze several tranches with different attachments, detachments, and maturities at once, it is more convenient to solve the forward version of equation (50) and find $p_{\alpha,\delta}(0, T)$ by integration. Thus, we are in a paradoxical situation when it is more efficient to perform calibration to individual names backward and calibration to tranches forward, rather than the other way round.

When derivatives explicitly depending on the number of defaults, such as leveraged super-senior (LSS) tranches, are considered, the X, Y dynamics requires augmentation with the dynamics of the number of defaulted names n. Since we are dealing with a "pure birth" process, we can use the well-known results due to Feller [5] and others and obtain the following expression for the one-step transition probability:

$$\begin{aligned} h(t, X, Y, n) &= \frac{-\sum_{n'=0}^{n}\left[p_t\left(t, T, n' \mid Y\right) + Xp_Y\left(t, T, n' \mid Y\right)\right]}{p\left(t, T, n \mid Y\right)} \\ &= \frac{\sum_{n'=n+1}^{N}\left[p_t\left(t, T, n' \mid Y\right) + Xp_Y\left(t, T, n' \mid Y\right)\right]}{p\left(t, T, n \mid Y\right)} \end{aligned} \tag{56}$$

The corresponding backward Kolmogoroff equation has the following form:

$$\hat{\mathcal{L}}V(t, X, Y, n) + XV_Y(t, X, Y, n) + h(t, x, Y, n) \times \left[V(t, X, Y, n+1) - V(t, X, Y, n)\right] = 0 \tag{57}$$

Table 1 Market quotes and full dynamic model calibration results. We quote par up-front payments with 5% spread for equity tranches, and par spreads for all other tranches[a]

α	δ	5y Market	5y Model	7y Market	7y Model	10y Market	10y Model
0%	3%	21.75%	21.76%	29.00%	28.89%	36.88%	36.94%
3%	6%	150.5	149.8	210.5	215.6	377.0	379.5
6%	9%	72.5	73.7	108.0	100.7	158.0	159.2
9%	12%	52.5	51.3	72.0	72.3	104.5	98.8
12%	22%	32.5	32.6	46.0	47.6	63.5	64.3
0%	100%	49.0	46.7	56.0	53.6	65.0	63.4

[a]adapted from [7]

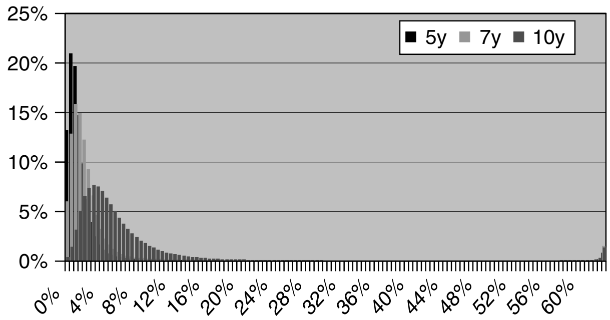

Figure 3 Loss distributions for 5y, 7y, 10y implied by the calibrated dynamic model (adapted from [7])

If need occurs, a multifactor extension of the above model can be considered.

Table 1 shows the quality of calibration achievable in the above framework for the on-the-run iTraxx index on November 9, 2007. We show the corresponding loss distributions in Figure 3.

This model can naturally be used to price bespoke baskets (as long as an appropriate standard basket is determined). It does not suffer from any of the drawbacks of the standard mapping approaches used for this purpose.

We note in passing that Inglis–Lipton [6] describe a static version of their model which is perfectly adequate for the purposes of pricing standard and bespoke tranches, even under the current extreme market conditions.

Conclusion

In general, multiname intensity models have many attractive features. They are naturally connected to single-name intensity models. In order to account for the observed tranche spreads in the market, they have to postulate periods of very high intensities which gradually mean-revert to moderate and low levels. Mean-reversion of the default intensities serves as a useful mechanism which allows one to price tranches with different maturities in a coherent fashion. Of course, due to the presence of large jumps, it is very difficult to provide convincing hedging mechanisms in such models. However, since we assume that jumps are discrete, it is possible in principle to hedge a given bespoke tranche with a *portfolio* of standard tranches. This is a topic of active research and experimentation at the moment, and we hope to present the outcome of this research in the near future.

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ALEXANDER LIPTON

Default Time Copulas

Copulas are used in mathematical statistics to describe *multivariate distributions* in a way that separates the marginal distributions from the codependence structure. More precisely, any multivariate distribution can be "decomposed" into its *marginal distributions* and a multivariate distribution with uniform marginals. Suppose $X_1, \ldots, X_n$ are real-valued stochastic variables with marginal distributions

$$f_i(x) = P(X_i \leq x), \quad i = 1, \ldots, n \tag{1}$$

where the right-hand side denotes the probability that X_i takes a value less than or equal to x. Suppose further that C is a distribution function on the n-dimensional unit hypercube with uniform marginals.[a] Then we can define a joint distribution of $(X_1, \ldots, X_n)$ by

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) = C(f_1(x_1), \ldots, f_n(x_n)) \tag{2}$$

We say that C is the *copula function* of the joint distribution. Clearly, the copula function for a given distribution is unique. Existence, that is, the actual existence of a copula function for any joint distribution, is established by *Sklar's Theorem* [3]. Given the definition of a copula, it is clear that a default time copula is a copula for the joint distribution of default times. Here, as in other applications in finance, the main advantage of using a copula formulation is that the marginal distributions are implied from the market, independent of information about mutual dependencies between default times. Specifically, the distribution of the time of default of a single firm can be implied[b] from the par spread of the credit default swap (CDS) contracts on the debt of the firm. This distribution is represented by the "default curve":

$$p_i(t) = P(\tau_i \leq t) \tag{3}$$

where τ_i is the stochastic default time of the ith firm.[c] Once we have determined the marginal distributions of the default rates of single firms in this way, we may model mutual dependencies between these default times by choosing a suitable copula function and writing the joint distribution of default times as in equation (2). From a practical point of view it is a great advantage that, by construction, the marginal distributions are unchanged under a change of copula. This allows us to preserve the calibration to market CDS quotes while adjusting the codependence structure.

Factor Copulas

In practice, copula functions are rarely specified directly for the default times. Instead, we introduce stochastic "default trigger variables" X_i such that we can identify events

$$\{X_i \leq h_i(t)\} \equiv \{\tau_i \leq t\} \tag{4}$$

for suitable nondecreasing functions $h_i : \mathbb{R}_+ \to \mathbb{R}$ such that

$$P(X_i \leq h_i(t)) = p_i(t) \tag{5}$$

We may regard the trigger variables as just a convenient mathematical device, but we may also follow Merton [2] and view X_i as the (return of the) value of the assets of the ith firm. With this interpretation, we may further interpret $h_i(T)$ for some fixed time horizon T as the face value of the firm's debt maturing at T. In this picture, default coincides with insolvency.

One advantage of using default trigger variables rather than default times is that the codependency of firm values is more susceptible to economic reasoning. For example, we can think of asset values as being driven by a common factor representing general economic conditions. Then we would use a decomposition such as

$$X_i = f_i(Z) + \epsilon_i \tag{6}$$

where Z is the common factor, ϵ_i are idiosyncratic components independent of each other and of Z, and f_i are suitable "loading functions". Note that, conditional on a given factor value, the trigger variables and, therefore, the default times, will be independent. The (unconditional) joint distribution is determined, for given distributions of Z and the ϵ_is, by the loading functions f_i.

A default time copula specified by default triggers with the decomposition in equation (6) is called a *factor copula*. Most, if not all, copula models used in derivatives pricing are factor copulas.

Pricing with Copula Models

The generic application of default time copulas is in the pricing of CDO tranches, that is, tranches of a portfolio of debt instruments referencing a (large) number of issuers. Such a tranche is a special case of a security whose future cash flows is a function of the default times of the issuers. The present value of such a security is given by an expectation over the joint default time distribution, which, in the general case, has to be evaluated by Monte Carlo, that is, by random sampling from the distribution. However, as we shall now discuss, for certain types of securities, the expectation can be calculated by a much faster method if a factor copula is used.

Loss Distributions

Although it is true that a CDO tranche depends on the joint default time distribution, it does so in a rather special way since, in fact, it only depends on the total *loss* in the portfolio; in particular, it does not depend on the identity of the defaulted names, or on the order in which they default. More precisely, we can compute the value of a tranche if we know the distribution of the cumulated portfolio loss out to any time up to tranche maturity.[d] As we shall now see, the computation of such loss distributions is particularly simple in a factor model.

We shall first show how to compute the distribution of portfolio loss to some fixed horizon t conditional on some given factor value z. To lighten the notation, we suppress the parameters z and t. Let p_i be the conditional probability that the ith issuer defaults and assume that the loss in default is given by some constant[e] u. Further define

$$P_l^{(n)} = P(L^{(n)} = lu) \tag{7}$$

where $L^{(n)}$ is the default loss from the first n issuers (in some arbitrary order).

Then we have the following recursion relation (see [1])

$$P_l^{(n+1)} = (1 - p_{n+1})P_l^{(n)} + p_{n+1}P_{l-1}^{(n)} \tag{8}$$

which allows us to build the loss distribution for any portfolio from the trivial case of the empty portfolio

$$P_l^{(0)} = \delta_{l,0} \tag{9}$$

From the conditional loss distributions, we obtain the unconditional loss distribution by integration[f] over z.

We remark that using equation (8) amounts to explicitly doing the convolution of the independent conditional loss distribution for each issuer in order to obtain the distribution of the portfolio loss. This convolution could also be done by Fourier techniques although this involves a somewhat greater computational burden. Note that by suitably inverting the convolution, one may compute the sensitivities of the tranche value to the parameters, for example, default probability, of each issuer. These are very important quantities in financial risk management.

Concluding Remarks

Models based on default time copulas are in widespread use for pricing and risk managing portfolio credit derivatives such as CDO tranches. The important special case of factor copulas combines the dual advantages of providing a clear economical interpretation of default time codependence and of allowing computationally efficient implementations.

The main practical limitation of copula models is that they are not *dynamic* models in the sense that they do not allow any conditioning on the future state of the world. This means that copula models cannot be reliably used, for example, in the pricing of options on tranches since here we have to be able to determine the distribution of the value of the underlying tranche conditioned on the state at option expiration time. To address such problems, we need a model that specifies the stochastic dynamics of a sufficient set of state variables. For example, we could specify the joint dynamics of all default intensities. Any such model would, of course, produce a joint default time distribution which would be describable by a copula and marginals. But this is not a one-to-one relationship since different dynamic models can produce the same copula. In this sense, the copula approach is more efficient for securities that depend only on the joint distribution of default times.

End Notes

[a] This simply means that $C : [0, 1]^n \to [0, 1]$ is nondecreasing in each argument, $C(0, \ldots, 0) = 0$, $C(1, \ldots, 1) = 1$, and that, for any i and any $y_i \in [0, 1]$,

$$\int_0^1 \mathrm{d}y_1 \ldots \int_0^1 \mathrm{d}y_{i-1} \int_0^1 \mathrm{d}y_{i+1} \ldots \int_0^1 \mathrm{d}y_n\, C(y_1, \ldots, y_n) = y_i.$$

[b] Given suitable assumptions about *recovery* in default.
[c] Note that this distribution is the so-called risk-neutral distribution,which differs from the real-world, or physical, distribution unless there is no *risk premium* associated with the risk of default.
[d] In practice, this is approximated by a finite set of times.
[e] This assumption is just for notational convenience; the extension to issuer specific, and possibly random, loss amounts is straightforward.
[f] If z has real dimension ≤ 3, a quadrature scheme can be used, otherwise Monte Carlo integration is more efficient.

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Related Articles

Copulas: Estimation; **Copulas in Econometrics**; **Copulas in Insurance**; **Exposure to Default and Loss Given Default**; **Gaussian Copula Model**; **Random Factor Loading Model (for Portfolio Credit)**; **Recovery Rate**.

JAKOB SIDENIUS

Gaussian Copula Model

Li [5] has introduced a copula function approach to credit portfolio modeling. In this approach, the author first introduces a random variable to denote the survival time for each credit and characterizes its properties using a density function or a hazard rate (*see* **Hazard Rate**). This allows us to move away from a one-period framework so that we could incorporate the term structure of default probabilities for each name in the portfolio. Then, the author introduces copula functions (*see* **Copulas: Estimation**) to combine information from all individual credits and further assumes a correlation structure among all credits. Mathematically, copula functions allow us to construct a joint distribution of survival times with given marginal distributions as specified by individual credit curves. This two-stage approach to forming a joint distribution of survival times has advantages. First, it incorporates all information on each individual credit. Second, we have more choices of different copula functions to form a good joint distribution to serve our purpose than if we assume a joint distribution of survival times from the start. While the normal copula function was used in [5] for illustration due to the simplicity of its economic interpretation of the correlation parameters and the relative ease of computation of its distribution function, the framework does allow use of other copula functions. We also discuss an efficient "one-step" simulation algorithm of survival times in the copula framework by exploring the mathematical property of copula functions in contrast to the period by period simulation as suggested earlier by others.

Default Information of a Single Name

To price any basket credit derivative structure, we first need to build a credit curve for each single credit in the portfolio, and then we need to have a default correlation model so that we can link all individual credits in the portfolio.

A credit curve for a company is a series of default probabilities to future dates. Traditionally, we use rating agency's historical default experience to derive this information. From a relative value trading perspective, however, we rely more on market information from traded assets such as risky bond prices, asset swap spreads, or, nowadays, directly the single-name term structure of default swap spreads to derive market implied default probabilities. These probabilities are usually called *risk neutral default probabilities*, which, in general, are much higher than the historical default probabilities for the rating class to which this company belongs. Mathematically, we use the distribution function of survival time to describe these probabilities. If we denote τ as an individual credit's survival time which measures the length of time from today to the time of default, we use $F(t)$ as the distribution function defined as follows:

$$F(t) = \Pr[\tau \leq t] = 1 - S(t) \qquad (1)$$

where $S(t)$ is called the *survival probability* up to time t. The marginal probabilities of defaults such as the ones over one-year periods, or hazard rates in continuous term, are usually called a *credit curve*. In general, for single-name default swap pricing, only a credit curve is needed in the same way as an interest rate curve is needed to price an interest rate swap.

Correlating Defaults through Copula Functions

Central to the valuation of the credit derivatives based on a credit portfolio is the default correlation. To put it in simple terms, default correlation measures the impact of one credit default on other credits. Intuitively, one would think of default correlation as being driven by some common macroeconomic factors. These factors tend to tie all industries into the common economic cycle, a sector-specific effect or a company-specific effect. From this angle, it is generally believed that default correlation is positive even between companies in different sectors. Within the same sector, we would expect companies to have an even higher default correlation since they have more commonalities. For example, overcapacity in the telecommunication industry after the internet/telecom bubble resulted in the default of numerous telecommunication and telephone companies. However, the sheer lack of default data means those assumptions are difficult to verify with any degree of certainty. Then we have to resort to an economic model to solve this problem.

From a mathematical point of view, we know the marginal distribution of survival time of each credit in the portfolio and we need to find a joint survival time distribution function such that the marginal distributions are the same as the credit curves of individual credits. This problem cannot be solved uniquely. There exist a number of ways to construct a joint distribution with known marginals. Copula functions, used in multivariate statistics, provide a convenient way to specify any joint distribution with given marginal distributions.

A copula function (*see* **Copulas: Estimation**) is simply a specification of how to use the univariate marginal distributions to form a multivariate distribution. For example, if we have N-correlated uniform random variables $U_1, U_2, \ldots, U_N$, then

$$C(u_1, u_2, \ldots, u_N) = \Pr\{U_1 < u_1, U_2 < u_2, \ldots, U_N < u_N\} \quad (2)$$

is the joint distribution function, which gives the probability that all of the uniforms are in the specified N-dimensional space cube. Using this joint distribution function C and N marginal distribution functions $F_i(t_i)$, which describe N credit curves, we form another function as follows: $C[F_1(t_1), F_2(t_2), \ldots, F_N(t_N)]$. It can be shown that this function is a distribution function for the N-dimensional random vector of survival times where, as desired, the marginal distributions are $F_1(t_1), F_2(t_2), \ldots, F_N(t_N)$; see [5]. So a copula function is nothing more than a joint distribution of uniform random variables from which we can build a joint distribution with a set of given marginals.

Then we need to solve two problems. First, which copula function should we use? Second, how do we calibrate the parameters in a copula function? Suppose we study a credit portfolio of two credits over a given period. The marginal default probabilities are given by the two credit curves constructed using market information or historical information. From an economic perspective, a company defaults when its asset falls below its liability. However, in the relative value trading environment, we know the default probability from the credit curve constructed using market information such as default swap spreads, asset swap spreads, or risky bond prices. Assume that there exists a standardized "asset return" X and a critical value x, and when $X \le x$ the company would default, that is,

$$\Pr[X_1 \le x_1] = \Phi(x_1) = q_1$$
$$\Pr[X_2 \le x_2] = \Phi(x_2) = q_2 \quad (3)$$

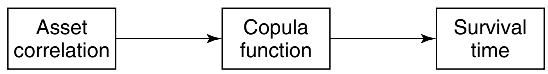
where Φ is the cumulative univariate standard normal distribution. We use Φ_n to denote the n-dimension cumulative normal distribution function. If we assume that the asset returns follow a bivariate normal distribution $\Phi_2(x, y, \rho)$ with correlation coefficient ρ, the joint default probability is given by

$$\begin{aligned}\Pr[X_1 \le x_1, X_2 \le r_2] &= \Pr[X_1 \le \Phi^{-1}(q_1), X_2 \le \Phi^{-1}(q_2)] \\ &= \Phi_2[\Phi^{-1}(q_1), \Phi^{-1}(q_2), \rho] \quad (4)\end{aligned}$$

This expression suggests that we can use a Gaussian copula function with asset return correlations as parameters.

The above argument need not be associated with a normal copula. Any other copula function would be still able to give us a joint survival time distribution while preserving the individual credit curves. We have to use extra conditions in order to choose an appropriate copula function. When we compare two copula functions, we need to control the marginal distribution-free correlation parameter such as the rank correlation.

This approach gives a very flexible framework based on which we can value many basket structures. It can be expressed in the following graph:



We also present an efficient simulation algorithm here to implement our framework. To simulate correlated survival times, we introduce another sequence of random variables $X_1, X_2, \ldots, X_n$ such that

$$X_i = \Phi^{-1}(F(\tau_i)) \quad (5)$$

where $\Phi^{-1}(\cdot)$ is a one-dimensional standard normal inverse function. $X_1, X_2, \ldots, X_n$ follow a joint normal distribution with a given correlation matrix Σ. From this equation, we see that there is a one-to-one mapping between X_i and τ_i. Any problem associated with τ_i could be transformed into a problem associated with X_i, which follows a joint normal distribution. Then we could make use of an efficient calculation method of multivariate normal distribution function.

The correlation parameters, in the framework of our credit portfolio model, can be roughly interpreted as the asset return correlation. However, in most practical uses of the current model, we either set the correlation matrix using one constant number or two numbers as the inter- and intraindustrial correlation for trading models. We could either use an economic model to asset correlation or we can calibrate the parameters using traded instruments involving correlation such as first-to-default or collateralized debt obligation (CDO) tranches.

The commonly used one or two correlation parameters are strongly associated with factor models for asset returns. For example, the one correlation parameter $\rho \geq 0$ corresponds to a one-factor asset return model where each asset return can be expressed as follows:

$$X_i = \sqrt{\rho} \cdot X_m + \sqrt{1 - \rho} \cdot X_i \tag{6}$$

where X_m represents the common factor return and ε_i is the idiosyncratic risk associated with credit asset i. Vasicek [7] and Finger [3] use this one-factor copula for portfolio loss calculation. For a detailed discussion on this one-factor copula model, the reader is referred to these two references.

If we use two parameters, the interindustrial correlation ρ_o and the intraindustry correlation ρ_I, then for each credit of industry group $k = 1, 2, \ldots, K$, we can express the asset return as follows [6]:

$$X_i = \sqrt{\rho_I - \rho_o \cdot} X_k + \sqrt{\rho_o} \cdot X_m + X_i \tag{7}$$

Using these factor models, we can substantially reduce the dimensionality of the model. The number of independent factors then does not depend on the size of the portfolio. For example, for a portfolio whose credits belong to 10 industries, we just need to use 11 independent factors, one factor for each industry and one common factor for all credits. We could substantially improve the efficiency of our simulation or analytical approach once we exploit the property of the factor models embedded in the correlation structure. Some other orthogonal transformations such as the ones obtained by applying principal component analysis could also be used to reduce the dimension.

Loss Distribution

For a given credit portfolio, the first information investors would like to know is its loss distribution over a given time horizon in the future such as 1 year or 5 years. This would give the investor some idea about the possible default loss of his investment in the next few years. The information we need to use in our framework is as follows: the credit curve of each credit that characterizes the default property over the time horizon, the recovery assumption, and the asset correlation structures. Many useful risk measurements, such as the expected loss, the unexpected loss, or the standard deviation of loss, the maximum loss, Value-at-Risk (VaR) or the conditional shortfall, could be obtained easily once the total loss distribution is calculated.

Here we study the property of the loss distribution using a numerical example. The base case used is as given in Table 1.

Figure 1 shows the excess loss distribution where the x-axis is the loss amount and y-axis is the probability of loss more than a given amount in the x-axis. All excess loss functions would start from 1 and gradually go to zero. If we include the zero loss in the probability calculation, then the probability of having nonnegative losses is always 1. We purposely exclude the zero loss in the calculation so that we can see the probability of having zero loss in the graph explicitly. Let us define the excess loss more precisely. Suppose that L represents the total loss of the portfolio, which is a random variable, since we do not know for sure what value it takes. For a given set of loss amounts $l_0, l_1, \ldots, l_n$, we can calculate the probability of excess loss $p_0, p_1, \ldots, p_n$ as follows:

$$p_i = S(l_i) = \Pr[L > l_i] \tag{8}$$

The excess loss distribution essentially depicts (l_i, p_i). The reason we use excess loss distribution instead of loss distribution, which is defined as $F(l_i) = 1 - S(l_i)$, is mainly due to the fact that many interesting properties of the loss distribution can be viewed more explicitly from the excess loss distribution graph than from the ordinary loss distribution

Table 1 Assumptions on a Credit Portfolio

Number of Assets	100
Credit spread	200 bps
Correlation	50%
Maturity	5 years
Recovery	30%

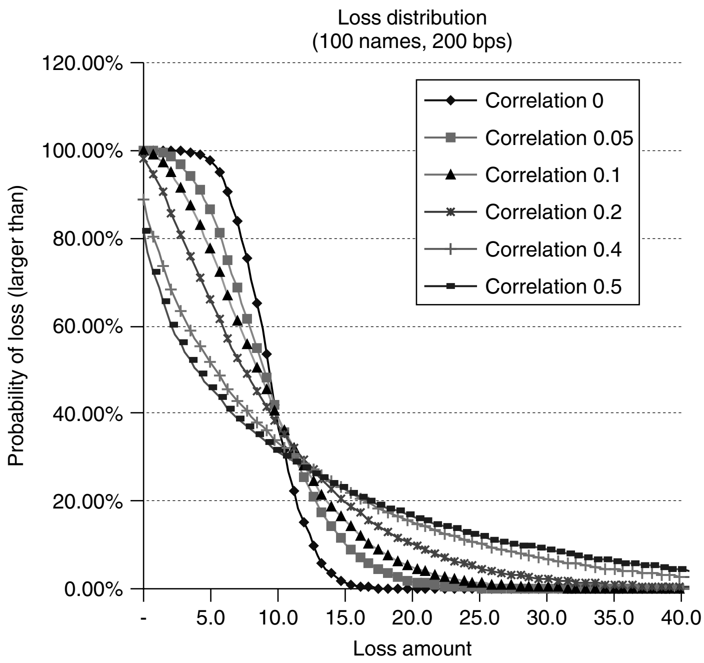


Figure 1 Excess Loss Distribution

graph. For example, the expected loss using the density function $f(l)$ of the loss distribution can be calculated as follows:

$$\mu_L = E(L) = \int_0^\infty l \cdot f(l)\mathrm{d}l = \int_0^\infty S(l)\mathrm{d}l \qquad (9)$$

which is just the area below the excess loss distribution line. Some other quantity such as the expected loss of tranched securities (loss with a deductible and a ceiling) could also be more simply expressed if we use excess loss function. We discuss this point in the next section when we discuss about CDO pricing.

Figure 1 shows the impact of correlation on the total excess loss distribution. From the graph we see that the probability of having zero loss increases from almost 0 to about 20% when correlation changes from 0 to 50%. The default probability over 5 years for each name is $1 - \mathrm{e}^{-5\cdot 0.2/(1-30\%)} = 13.31\%$, and the probability of having no default of a portfolio with 100 independent names is $(1 - 13.31\%)^{100}$, practically 0. However, when the correlation is high, default occurs more in bulk, which makes the probability of having zero loss go up to 20%. When correlation is high, more loss would be pushed to the right, which makes the excess loss distribution tail much fatter since the expected total loss, the area below the excess loss function line, does not change along with the change in correlation. This can be shown using a credit VaR, which is defined as the excess loss, that probability of loss larger than this value is less than a given percentage such as 1%. The 1% credit VaR for various correlation values are given in Table 2.

In practice, it is very important to quickly obtain an accurate total excess loss distribution. There are a variety of methods that have been used for the total loss distribution. Here, we present the details on the recursive method in a one-factor Gaussian copula model and briefly summarize the conditional normal approximation approach.

Table 2 Correlation vs C-VaR

Correlation (%)	C-VaR
0	14.7
10	25.2
20	32.9
50	53.9
75	67.2%

We consider a credit portfolio consisting of n underlying credits whose notional amounts are N_i and fixed recovery rates are R_i, $i = 1, 2, \ldots, n$. We consider the aggregate loss from today to time t as a fixed sum of random variables X_i:

$$L_n(t) = \sum_{i=1}^{n} l_i(t) = \sum_{i=1}^{n} (1 - R_i) \cdot N_i \cdot I_{(\tau_i < t)} \tag{10}$$

where τ_i is the survival time for the ith credit in the credit portfolio and I is the indicator function, which is 1 in the case $\tau_i \leq t$ and 0 otherwise. The distribution function of survival time c is denoted as $F_i(t) = \Pr[\tau_i \leq t]$. The specification of the survival time distribution $F_i(t)$ is usually called a *credit curve*, which can be derived from market credit default swap spreads.

From the above equation, we can calculate the total loss distribution as

$$\begin{aligned} F_{L(t)}(x) = \Pr[L_n(t) \leq x] &= \Pr\left[\sum_{i=1}^{n} X_i \leq x\right] \\ &= \int \Pr\left[\sum_{i=1}^{n} X_i \leq x | F\right] \cdot \mathrm{d}F \end{aligned} \tag{11}$$

Conditional on the common factor F, all X_i are independent, and then we just need to calculate the convolution of n independent random variables, c. As discussed in the last section, we know that X_i are independent conditional on the common factor X_M in the one-factor model. Each X_i can take only two discrete values with constant recovery rate assumption as follows: the loss would be 0 if default does not occur or $B_i = (1 - R_i)N_i$ if default occurs.

$$f(x|F) = \begin{cases} 0, & 1 - q_i(t|F) \\ B_i, & q_i(t|F) \end{cases} \tag{12}$$

where $q_i(t|F)$ is the conditional default probability for credit i before time t.

The density of the conditional total loss distribution can be calculated recursively over the partial sum $L_j = L_{j-1} + X_j$. We then have the following recursive formula:

$$f_{L_j}(x|F) = \begin{cases} p_j \cdot f_{L_{j-1}}(x|F), & x < B_j \\ p_j \cdot f_{L_{j-1}}(x|F) + q_j \cdot f_{L_{j-1}}(x - B_j|F), & x \geq B_j \end{cases} \tag{13}$$

This has been described in [4] and also in [1]. The unconditional total loss distribution is obtained by simply integrating the conditional loss distribution over the common factor F. In the simple case of one-factor Gaussian copula model, we use a Gaussian quadrature for the integration over the one common factor. In the one-parameter case, the conditional default probability can be calculated directly as follows:

$$\begin{aligned} q_i(t|X_M) &= \Pr[\tau_i < t|X_M] \\ &= \Pr[F_i^{-1}(N(X_i)) < t|X_M] \\ &= \Pr[X_i < N^{-1}(F(t))|X_M] \\ &= N\left(\frac{N^{-1}(q_i(t)) - \rho \cdot X_M}{\sqrt{1-\rho}}\right) \end{aligned} \tag{14}$$

where $q_i(t)$ is the unconditional default probability of credit i before time t.

Another approach uses the conditional normal approximation. Conditional on the common factors, all credits are independent. On the basis of the law of large numbers, the total conditional loss distribution can be approximated by a normal distribution. The mean and variance of this normal distribution can be simply calculated similarly as we do in the above one-factor case. More details are given as follows.

Conditioning on the common factor X_M, we can compute the mean and variance of the total loss variable, $L|X_M$,

$$\begin{aligned} M_v &= \sum_{i=1}^{n} N_i \cdot (1 - R_i) \cdot q_i\,(t|X_M) \\ \sigma_v^2 &= \sum_{i=1}^{n} N_i^2 \cdot (1 - R_i)^2 \cdot q_i(t|X_M)(1 - q_i(t|X_M)) \end{aligned} \tag{15}$$

The conditional normal approach uses normal distributions to approximate the conditional loss distribution. The normal distribution has the same mean and variance as computed above. In general, other distributions, such as inverse normal or Student-t, can be used. The normal distribution is chosen becuase of the central limit theorem, which states that the sum of independent distributions (but not identical distribution) approaches a normal distribution as the number of the independent distributions increases. In this case, the independent distributions are the distributions of the indicator functions of $N_i \cdot (1 - R_i) \cdot l_{[\tau_i < t]}$, which are independent when conditioned on the common factor X_M.

Given the conditional normal approach, the conditional expected loss for a tranche with attachment and detachment K_L^T and K_U^T can be easily computed in closed form as follows:

$$E(L^T(t)|X_M) = (M_v - K_L^T)\Phi\left(\frac{M_v - K_L^T}{\sigma_v}\right) + \sigma_v \cdot \phi\left(\frac{M_v - K_L^T}{\sigma_v}\right) - (M_v - K_U^T)\Phi\left(\frac{M_v - K_U^T}{\sigma_v}\right) - \sigma_v \cdot \phi\left(\frac{M_v - K_U^T}{\sigma_v}\right) \quad (16)$$

where ϕ is the one-dimensional normal density function.

With the calculated conditional expected loss, the unconditional expected loss is obtained simply by integrating over the common factor X_M.

$$E(L^T(t)) = \int_{-\infty}^{+\infty} E(L^T(t)|y) \cdot \phi(y)\mathrm{d}y \quad (17)$$

However, by choosing normal distribution in its approximation, the approach has its limitations. First, a normal variable can have a negative value with no-zero probability. As we know, the loss in a portfolio should never be negative. However, this limitation only affects the equity tranche (the most junior tranche) and can be mitigated through the method described below. Second, as a loss is a summation of discrete loss variables, when a portfolio consists of only a few underlying names, then approximating the loss by a continuous variable (such as a normal variable) might not be a good approximation. This limitation also applies to some extreme case when the loss is dominated by only a few underlying names. In general, the conditional normal approach is a very good approximation when the number of names in a portfolio is larger than 30. Most of the CDO portfolios have the number of names larger than 30.

To mitigate the negative loss problem for an equity tranche one can use the following method, which preserves the expected loss of a CDO portfolio. An equity tranche $[0, K_U^T]$ with detachment point K_U^T has payoff as follows:

$$L^T(t) = L(t) - \max[L(t) - K_U^T, 0] \quad (18)$$

The conditional expected loss for the equity tranche is

$$E(L^T(t)|X_M) = M_v - (M_v - K_U^T) \cdot \Phi\left(\frac{M_v - K_U^T}{\sigma_v}\right) - \sigma_v \cdot \phi\left(\frac{M_v - K_U^T}{\sigma_v}\right) \quad (19)$$

This has been proven to work well for index equity tranche of size more than 2%. Alternatively, we can also use inverse Gaussian distribution to approximate for the equity tranche since inverse Gaussian distribution takes only a positive value.

Risk Measurement and Hedging

Once a model and a mapping algorithm are chosen we can price all credit portfolio trades, and produce a series of risk measures based on the model. These risk measures are then used to form a hedging strategy for the trading book. The commonly used risk measures are as follows:

- Credit spread delta: This is defined as the sensitivity of the mark-to-market value of a position to the instantaneous movement of the spread of a single entity, with all other parameters remaining constant. It is calculated through perturbation of the individual credit curves. Individual spread delta is reported as the change in value of the trade for a 1 basis point (1 bp) move in the indicated spread. Individual spread delta can be calculated as parallel moves in the individual curve (in which each spread on a particular curve is moved by 1 bp in a parallel fashion) or as bucketed moves in the curve. Individual spread delta is calculated trade-by-trade, and aggregated on the basis of issuer name, industry, or portfolio level. When we change the spread, we recalibrate the credit curve or instantaneous marginal default probability. Global spread delta is defined as the change in the portfolio value change when all the underlying reference credit curves move by 1 bp. Global spread is calculated by bumping all spread curves of the underlying reference credits simultaneously in a parallel way or in buckets. Sometimes, we also study the sensitivities of our trade or book with respect to a large spread movement. Another common practice is to adjust the individual spread movement with respect to the index spreadsheet. The reason is that not all individual spread moves by the same amount when index moves. A statistical beta based on regression analysis is usually used.

- Single-name spread gamma: This is defined as the sensitivity of individual spread delta to a 1-bp move in a particular reference credit. As such, it represents the second-order price sensitivity with respect to a change in the spreads of the reference credit. Individual spread gamma is calculated by bumping one credit curve a time while all other credit curves remain the same for portfolio transactions. Global gamma is defined as the change in the global spread delta (which is defined as the portfolio value change when all the underlying reference credit curves move by 1 bp) of a portfolio for a 1-bp move in all reference credit spreads simultaneously. Global spread gamma is calculated by bumping all spread curves of the underlying reference credits simultaneously in a parallel way. Sometimes, we simply use a large spread movement as a measure of gamma risk by bumping the current spread by 50%. Similar to spread delta risk, we can also use bucket gamma risks which are more computationally challenging.
- Jump-to-default risk: We measure it by simply assuming one-name defaults right away or at a specific time in the future. We can also study the group jump-to-default risk.
- Time-decay risk: This measures the risk that as time passes, or maturity shortens, the value of portfolio transactions changes. For portfolio credit default swap, its survival time curve is most likely not flat, which makes the time decay an important risk factor.
- Correlation risk: Since we use a base correlation curve, we could measure the risk in terms of parallel change or bucket correlation change. In practice, we very often see a correlation curve twist, which reflects market changing perception about different tranche risks. We can measure this risk by creating a sensitivity report for the whole book with respect to each base correlation point.

In practice, we tend to minimize spread and gamma risks, control jump-to-default risk, and also make correlation risk flat. We would also like to have positive carry: we receive more cash inflow than outflow. The hedging instruments we use are single name and index credit default swaps and index tranches. Very often, broker dealers tend to incur residual risks by using hedge ratios higher than the model-based amount to maintain a positive carry, but this strategy does not work well all the time, especially during turmoil, when there are unexpected defaults or jumps in spreads. We can use index tranche to hedge the base correlation risk. Sometimes, we can also use the index plus complementary tranches to hedge the correlation risk. In conclusion, for any hedging strategy, there will be a residual risk. Traders very often use their own view toward the market to selectively keep some residual risk.

In conclusion, the Gaussian copula function approach along with a base correlation method provides a simple and flexible framework to price basket credit derivatives. We further studied the framework and gained some more insights of it, especially from the conditional perspective of its correlation structure. This shows that the Gaussian copula function implies a too strong correlation structure. The reason for this is that we describe each credit using only two states: default or survival. This simple way of binary description creates too strong a conditional default property. It is also associated with the simple way of specifying the correlation structure using only one parameter or pairwise constant correlation, in practice, even though the original framework allows a completely flexible correlation matrix specification. Another possible reason is that this framework still misses certain fundamental driving factors such as volatilities of individual names in the framework.

We briefly discuss the risk measurement and risk management issues using the Gaussian copula function method. From the pricing formula, we can obtain all necessary risk measures, such as spread DV01, jump-to-default risk, and gamma risk of individual spreads or the general index. We can also obtain the sensitivities of the portfolio of portfolio transactions with respect to each point of base correlation curve.