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Chapter 1

Fuzzy Sets

This chapter begins with a brief review of classical sets in order to facilitate the introduction of fuzzy sets. Next the concept of membership function is explained. It defines the degree to which an element under consideration belongs to a fuzzy set. Fuzzy numbers are described as a particular case of fuzzy sets. Fuzzy sets and fuzzy numbers will be used in fuzzy logic to model words such as profit, investment, cost, income, age, etc. Fuzzy relations together with some operations on fuzzy relations are introduced as a generalization of fuzzy sets and ordinary relations. They have application in database models. Fuzzy sets and fuzzy relations play an important role in fuzzy logic.

1.1 Classical Sets: Relations and Functions

Classical sets

This section reviews briefly the terminology, notations, and basic properties of *classical sets*, usually called *sets*.

The concept of a *set* or *collection* of objects is common in our everyday experience. For instance, all persons listed in a certain telephone directory, all employees in a company, etc. There is a defining property that allows us to consider the objects as a whole. The objects in a set are called *elements* or *members* of the set. We will denote elements by small letters $a, b, c, \ldots, x, y, z$ and the sets by capital letters

$A, B, C, \dots, X, Y, Z$. Sets are also called ordinary or crisp in order to be distinguished from fuzzy sets.

The fundamental notion in set theory is that of *belonging* or *membership*. If an object x belongs to the set A we write $x \in A$; if x is not a member of A, we write $x \notin A$. In other words for each object x there are only two possibilities: either x belongs to A or it does not.[1]

A set containing finite number of members is called *finite* set; otherwise it is called *infinite* set. We present two methods of describing sets.

Listing method

The set is described by *listing* its elements placed in braces; for example $A = \{1, 3, 6, 7, 8\}$, $B = \{\text{business, finance, management}\}$. The order in which elements are listed is of no importance. An element should be listed only once.

Membership rule

The set is described by one or more properties to be satisfied only by objects in the set:

$$A = \{x \mid x \text{ satisfies some property or properties}\}.$$

This reads: "A is the set of all x such that x satisfies some property or properties." For example $R = \{x \mid x \text{ is real number}\}$ reads: "R is the set of all x such that x is a real number"; $R_+ = \{x | x \geq 0, x \in R\}$ reads "R_+ is the set of all x which are nonnegative real numbers."

Universal set

The set of all objects under consideration in a particular situation is called *universal set or universe*; it will be denoted by U.

Empty set

A set without elements is called empty; it is denoted by ϕ.

Interval

The set of all real numbers x such that $a_1 \leq x \leq a_2$, where a_1 and a_2 are real numbers, form a closed interval $[a_1, a_2] = \{x \mid a_1 \leq x \leq a_2, x \in R\}$ with boundaries a_1 and a_2. It is also called *interval number*.

Equal sets

Sets A and B are *equal* , denoted by $A = B$, if they have the same elements.

Subset

The set A is a *subset* of the set B (A is *included* in B), denoted by $A \subseteq B$, if every element of A is also an element of B. Every set is subset of itself, $A \subseteq A$. The empty set ϕ is a subset of any set. It is assumed that each set we are dealing with is a subset of a universal set U.

Proper subset

A is a *proper subset* of B, denoted $A \subset B$, if $A \subseteq B$ and there is at least one element in B which does not belong to A. For instance $\{a, b\} \subset \{a, b, c\}$. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Intersection

The intersection of the sets A and B, denoted by $A \cap B$, is defined by

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}; \tag{1.1}$$

$A \cap B$ is a set whose elements are common to A and B.

Union

The union of A and B , denoted by $A \cup B$, is defined by

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}; \tag{1.2}$$

$A \cup B$ is a set whose elements are in A or B, including any element that belongs to both A and B.

Disjoint sets

If the sets A and B have no elements in common, they are called *disjoint*.

Complement

The *complement* of $A \subset U$, denoted by $\overline{A}$, is the set

$$\overline{A} = \{x \in U \mid x \notin A\}. \tag{1.3}$$

The complement of a set consists of all elements in the universal set that are not in the given set.

Example 1.1

Given the sets

$$A = \{1, 2, 3, 4\}, \quad B = \{1, 3, 5, 6\}, \quad U = \{1, 2, 3, 4, 5, 6, 7\},$$

then using (1.1)–(1.3) we find

$$A \cap B = \{1, 3\}, \;\; A \cup B = \{1, 2, 3, 4, 5, 6\}, \;\; \overline{A} = \{5, 6, 7\}, \;\; \overline{B} = \{2, 4, 7\}.$$

□

Convex sets

Consider the universe U to be the set of real numbers R.

A subset S of R is said to be *convex* if and only if, for all $x_1, x_2 \in S$ and for every real number λ satisfying $0 \leq \lambda \leq 1$, we have

$$\lambda x_1 + (1 - \lambda) x_2 \in S.$$

For example, any interval $S = [a_1, a_2]$ is a convex set since the above condition is satisfied; [0, 1] and [3, 4] are convex, but $[0, 1] \cup [3, 4]$ is not.

Venn diagrams

Sets are geometrically represented by circles inside a rectangle (the universal set U). In Fig. 1.1 are shown the sets $A \cap B$ and $A \cup B$.

Fig. 1.1. Venn diagrams for $A \cap B$(intersection), $A \cup B$(union).

Ordered pairs

It was noted that the order of the elements of a set is not important. However there are cases when the order is important. To indicate that a set or pair of two elements a and b is *ordered*, we write (a, b), i.e. use parentheses instead of braces; a is called *first element* of the pair and b is called *second element*.

Cartesian product

Cartesian product (or *cross product*) of the sets A and B denoted $A \times B$ is the set of ordered pairs

$$A \times B = \{(a, b) \mid a \in A, b \in B\}. \tag{1.4}$$

Example 1.2

(a) Given

$$A = \{1, 2, 3\}, \qquad B = \{1, 2\},$$

then according to (1.4) we find

$$A \times B = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)\};$$

geometrically it is presented on Fig. 1.2 (a).
(b) If $X, Y = R$, the set of all real numbers, then

$$X \times Y = \{(x, y) | x \in X, y \in Y\} = R \times R$$

is the set of all ordered pairs which form the cartesian plane xy (see Fig. 1.2(b)).

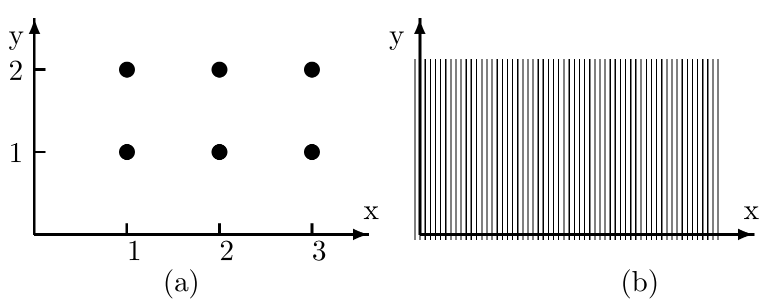

Fig. 1.2. (a) Cartesian product $\{1, 2, 3\} \times \{1, 2\}$; (b) Cartesian plane.

□

Relations

The concept of *relation* is very general. It is based on the concepts of ordered pair (a, b), $a \in A$, $b \in B$, and cartesian product of the sets A and B.

A *relation* from A to B (or between A and B) is any subset $\Re$ of the cartesian product $A \times B$. We say that $a \in A$ and $b \in B$ are related by $\Re$; the elements a and b form the *domain* and *range* of the relation, correspondingly. Since a relation is a set, it may be described by either the listing method or the membership rule. The relation $\Re$ is called *binary relation* since two sets, A and B, are related.

Example 1.3

Let $A = \{x_1, x_2, x_3\}$ and $B = \{1, 2, 3, 4\}$.
We list some binary relations generated by A and B:

$$\begin{aligned}
&\Re_1 = \{(x_1, 1), (x_2, 1), (x_3, 4)\},\\
&\Re_2 = \{(x_1, 2), (x_1, 3)\}, \quad \Re_3 = \{(x_2, 2), (x_3, 1)\},\\
&\Re_4 = \{(x_1, 1), (x_1, 2), (x_1, 3), (x_1, 4), (x_2, 1), (x_4, 1)\}
\end{aligned}$$

are relations from A to B;

$$\begin{aligned}
&\Re_5 = \{(1, x_2), (2, x_3), (3, x_1)\}, \quad \Re_6 = \{(1, x_1), (2, x_1)\},\\
&\Re_7 = \{(1, x_1), (1, x_2), (1, x_4)\}, \quad \Re_8 = \{(2, x_1), (3, x_3)\}
\end{aligned}$$

are relations from B to A; the empty set ϕ is a relation; the cross product $A \times B$ is a relation from A to B and the cross product $B \times A$ is a relation from B to A.

□

Functions

A *function* f is a relation $\Re$ such that for every element x in the domain of f there corresponds a unique element y in the range of f. For instance the relations in Example 1.2 are not functions.

We often say that f *maps* x onto y; y is the *image* of x under f. Then we can write $f : x \to y$. However, it is customary to use the notation $y = f(x)$.

Generalization

The notions of ordered pair, Cartesian product, relation, and function can be generalized for higher dimensions than two. For instance when $n = 3$ we have:

Ordered triple (a, b, c);

Cartesian product

$$A \times B \times C = \{(a, b, c) | a \in A, b \in B, c \in C\};$$

Relation from $A \times B$ to C is any subset $\Re$ of $A \times B \times C$.

Function $z = f(x, y)$ is a relation such that for every pair (x, y) in the domain of f there corresponds a unique element z in its range.

Characteristic Function

The membership rule that characterizes the elements (members) of a set $A \subset U$ can be established by the concept of *characteristic function* (or *membership function*) $\mu_A(x)$ taking only two values, 1 and 0, indicating whether or not $x \in U$ is a member of A:

$$\mu_A(x) = \begin{cases} 1 & \text{for} \quad x \in A, \\ 0 & \text{for} \quad x \notin A. \end{cases} \tag{1.5}$$

Hence $\mu_A(x) \in \{0, 1\}$. Inversely, if a function $\mu_A(x)$ is defined by (1.5), then it is the characteristic function for a set $A \subset U$ in the sense that A consists of the values of $x \in U$ for which $\mu_A(x)$ is equal to 1. In other words every set is uniquely determined by its characteristic function.

The universal set U has for membership function $\mu_U(x)$ which is identically equal to 1, i.e. $\mu_U(x) = 1$. The empty set ϕ has for membership function $\mu_\phi(x) = 0$.

Example 1.4

Consider the universe $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and its subset A,

$$A = \{x_2, x_3, x_5\}.$$

Only three of the six elements in U belong A. Using the notation (1.5) gives

$$\mu_A(x_1) = 0, \;\; \mu_A(x_2) = 1, \;\; \mu_A(x_3) = 1,$$
$$\mu_A(x_4) = 0, \;\; \mu_A(x_5) = 1, \;\; \mu_A(x_6) = 0.$$

Hence the characteristic function of the set A is

$$\mu_A(x) = \begin{cases} 1 & \text{for} \quad x = x_2, x_3, x_5, \\ 0 & \text{for} \quad x = x_1, x_4, x_6; \end{cases}$$

The set A can be represented as

$$A = \{(x_1, 0), (x_2, 1), (x_3, 1), (x_4, 0), (x_5, 1), (x_6, 0)\}.$$

□

Example 1.5

Let us try to use crisp sets to describe *tall men*. Consider for instance a man as tall if his height is 180 cm or greater; otherwise the man is not tall. The characteristic function of the set $A = \{\text{tall men}\}$ then is

$$\mu_A(x) = \begin{cases} 1 & \text{for} \quad 180 \leq x, \\ 0 & \text{for} \quad 160 \leq x < 180. \end{cases}$$

It is shown in Fig. 1.3, where the universe is $U = \{x \mid 160 \leq x \leq 200\}$.

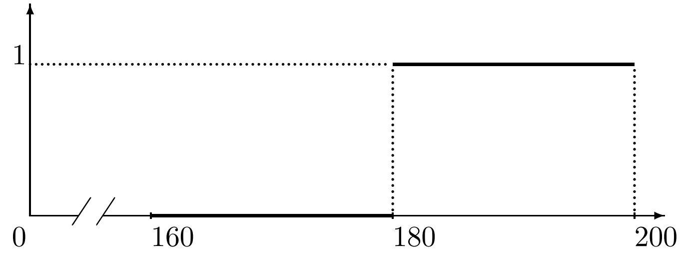

Fig. 1.3. Membership function of the set *tall men*.

Clearly this description of the set of *tall men* is not satisfactory since it does not allow gradation. The word *tall* is vague. For instance, a person whose height is 179 cm is not tall as well as a person whose height is 160 cm. Yet a person whose height is 180 is tall and so is a person with height 200 cm. Also the above definition introduces a drastic difference between heights of 179 cm and 180 cm, thus fails to describe realistically borderline cases.[2]

□

The concept of characteristic function introduced here will facilitate the understanding of the concept *fuzzy set*, the subject of the next section.

1.2 Definition of Fuzzy Sets

We have seen that belonging or membership of an object to a set is a precise concept; the object is either a member to a set or it is not, hence the membership function can take only two values, 1 or 0. The set *tall men* in Example 1.5 illustrates the need to increase the describing capabilities of classical sets while dealing with words.

To describe gradual transitions Zadeh (1965), the founder of fuzzy sets, introduced grades between 0 and 1 and the concept of graded membership.

Let us refer to Example 1.4. Each of the six elements of the universal set $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ either belongs to or does not belong to the set $A = \{x_2, x_3, x_5\}$. According to this, the characteristic function $\mu_A(x)$ takes only the values 1 or 0. Assume now that a characteristic function may take values in the interval [0, 1]. In this way the concept of membership is not any more *crisp* (either 1 or 0), but becomes *fuzzy* in the sense of representing partial belonging or *degree of membership.*

Consider a classical set A of the universe U. A *fuzzy set* $\mathcal{A}$ is defined by a set or ordered pairs, a binary relation,

$$\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x)) \mid x \in A, \mu_{\mathcal{A}}(x) \in [0, 1]\}, \tag{1.6}$$

where $\mu_{\mathcal{A}}(x)$ is a function called *membership function*; $\mu_{\mathcal{A}}(x)$ specifies the *grade* or *degree* to which any element x in A belongs to the fuzzy set $\mathcal{A}$. Definition (1.6) associates with each element x in A a real number $\mu_{\mathcal{A}}(x)$ in the interval [0, 1] which is assigned to x. Larger values of $\mu_{\mathcal{A}}(x)$ indicate higher degrees of membership.[3]

Let us express the meaning of (1.6) in a slightly modified way. The first elements x in the pair $(x, \mu_{\mathcal{A}}(x))$ are given numbers or objects of the classical set A; they satisfy some property (P) under consideration partly (to various degrees). The second elements $\mu_{\mathcal{A}}(x)$ belong to the interval (classical set) $[0, 1]$; they indicate to what extent (degree) the elements x satisfy the property P.

It is assumed here that the membership function $\mu_{\mathcal{A}}(x)$ is either piecewise continuous or discrete.

The fuzzy set $\mathcal{A}$ according to definition (1.6) is formally *equal* to its membership function $\mu_{\mathcal{A}}(x)$. We will *identify* any fuzzy set with its membership function and use these two concepts as *interchangeable*. Also we may look at a fuzzy set over a domain A as a function mapping A into [0, 1].

Fuzzy sets are denoted by italic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ and the corresponding membership functions by $\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x), \mu_{\mathcal{C}}(x), \ldots$.

Elements with zero degree of membership in a fuzzy set are usually not listed.

Classical sets can be considered as a special case of fuzzy sets with all membership grades equal to 1.

A fuzzy set is called *normalized* when at least one $x \in A$ attains the maximum membership grade 1; otherwise the set is called *nonnormalized*. Assume the set $\mathcal{A}$ is nonnormalized; then $\max \mu_{\mathcal{A}}(x) < 1$. To normalize the set $\mathcal{A}$ means to normalize its membership function $\mu_{\mathcal{A}}(x)$, i.e. to divide it by $\max \mu_{\mathcal{A}}(x)$, which gives $\frac{\mu_{\mathcal{A}}(x)}{\max \mu_{\mathcal{A}}(x)}$.

$\mathcal{A}$ is called empty set labeled ϕ if $\mu_{\mathcal{A}}(x) = 0$ for each $x \in A$.

The fuzzy set $\mathcal{A} = \{(x_1, \mu_{\mathcal{A}}(x_1))\}$, where x_1 is the only value in $A \subset U$ and $\mu_{\mathcal{A}}(x_1) \in [0, 1]$, is called *fuzzy singleton*.

While the set A is a subset of the universal set U which is crisp, the fuzzy set $\mathcal{A}$ is not.

Instead of (1.6), some authors use the notation

$$\mathcal{A} = \{\mu_{\mathcal{A}}(x)/x, x \in A, \mu_{\mathcal{A}}(x) \in [0, 1]\},$$

where the symbol / is not a division sign but indicates that the top number $\mu_{\mathcal{A}}(x)$ is the membership value of the element x in the bottom.

Example 1.6

Consider the fuzzy set

$$\mathcal{A} = \{(x_1, 0.1), (x_2, 0.5), (x_3, 0.3), (x_4, 0.8), (x_5, 1), (x_6, 0.2)\}$$

which also can be represented as

$$\mathcal{A} = 0.1/x_1 + 0.5/x_2 + 0.3/x_3 + 0.8/x_4 + 1/x_5 + 0.2/x_6;$$

it is a discrete fuzzy set consisting of six ordered pairs. The elements $x_i, i = 1, \ldots, 6$, are not necessary numbers; they belong to the classical set $A = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ which is a subset of a certain universal set U. The membership function $\mu_{\mathcal{A}}(x)$ of $\mathcal{A}$ takes the following values on [0, 1]:

$$\begin{array}{lll} \mu_{\mathcal{A}}(x_1) = 0.1, & \mu_{\mathcal{A}}(x_2) = 0.5, & \mu_{\mathcal{A}}(x_3) = 0.3, \\ \mu_{\mathcal{A}}(x_4) = 0.8, & \mu_{\mathcal{A}}(x_5) = 1, & \mu_{\mathcal{A}}(x_6) = 0.2. \end{array}$$

The following interpretation could be given to $\mu_{\mathcal{A}}(x_i), i = 1, \cdots, 6$. The element x_5 is a *full* member of the fuzzy set $\mathcal{A}$, while the element x_1 is a member of $\mathcal{A}$ a *little* ($\mu_A(x_1) = 0.1$ is near 0); x_6 and x_3 are a *little more* members of $\mathcal{A}$; the element x_4 is *almost* a full member of $\mathcal{A}$, while x_2 is *more or less* a member of $\mathcal{A}$.

The fuzzy set $\mathcal{A}$ can be given also by the table

$$\mathcal{A} \triangleq \begin{array}{|cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline 0.1 & 0.5 & 0.3 & 0.8 & 1 & 0.2 \end{array}$$

where the symbol $\triangleq$ means "is defined by."

Now we specify in two different ways the elements x_i in A:

(a) Assume that $x_i, i = 1, \cdots, 6$, are integers, namely, $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5, x_6 = 6$; they belong to the set $A = \{1, 2, 3, 4, 5, 6\}$, a subset of the universe $U = N$, the set of all integers. The fuzzy set $\mathcal{A}$ becomes

$$\mathcal{A} = \{(1, 0.1), (2, 0.5), (3, 0.3), (4, 0.8), (5, 1), (6, 0.2)\};$$

its membership function $\mu_{\mathcal{A}}(x)$ shown in Fig. 1.4 by dots is a discrete one.

(b) Assume now that $x_i, i = 1, \ldots, 6$, are friends of George whose names are as follows: x_1 is Ron, x_2 is Ted, x_3 is John, x_4 is Joe, x_5 is Tom, and x_6 is Sam. They form a set of friends of George,

$$A = \{\text{Ron, Ted, John, Joe, Tom, Sam}\},$$

a subset of the universe U (all friends of George). The fuzzy set $\mathcal{A}$ here expresses *closeness of friends* of George on $A \subseteq U$:

$$\mathcal{A} = \{(\text{Ron}, 0.1), (\text{Ted}, 0.5), (\text{John}, 0.3), (\text{Joe}, 0.8), (\text{Tom}, 1), (\text{Sam}, 0.2)\}.$$

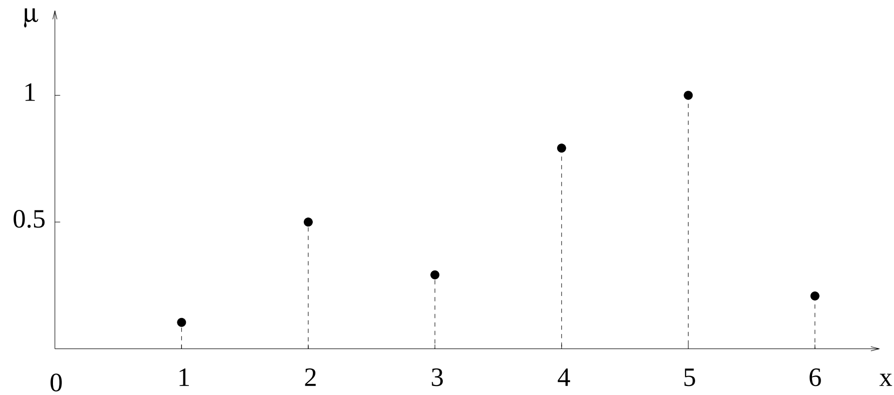

Fig. 1.4. Fuzzy set $\mathcal{A} = \{(1, 0.1), (2, 0.5), (3, 0.3), (4, 0.8), (5, 1), (6, 0.2)\}$.

□

Example 1.7

Let us describe numbers *close* to 10.

(a) First consider the fuzzy set

$$\mathcal{A}_1 = \{(x, \mu_{\mathcal{A}_1}(x)) \mid x \in [5, 15], \mu_{\mathcal{A}_1}(x) = \frac{1}{1 + (x - 10)^2}\},$$

where $\mu_{\mathcal{A}_1}(x)$ shown in Fig. 1.5 is a continuous function.

The fuzzy set $\mathcal{A}_1$ represents real numbers *close* to 10.

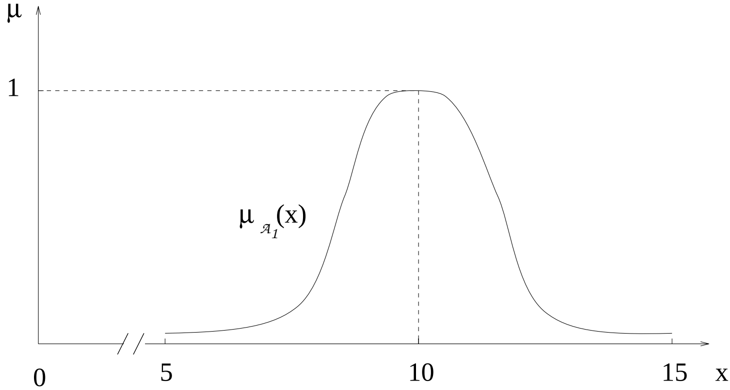

Fig. 1.5. Real numbers *close* to 10.

(b) Integers *close* to 10 can be expressed by the finite fuzzy set consisting of seven ordered pairs

$$\mathcal{A}_2 = \{(7, 0.1), (8, 0.3), (9, 0.8), (10, 1), (11, 0.8), (12, 0.3), (13, 0.1)\}.$$

The membership function of $\mathcal{A}_2$ is shown on Fig 1.6 by dots; it is a discrete function.

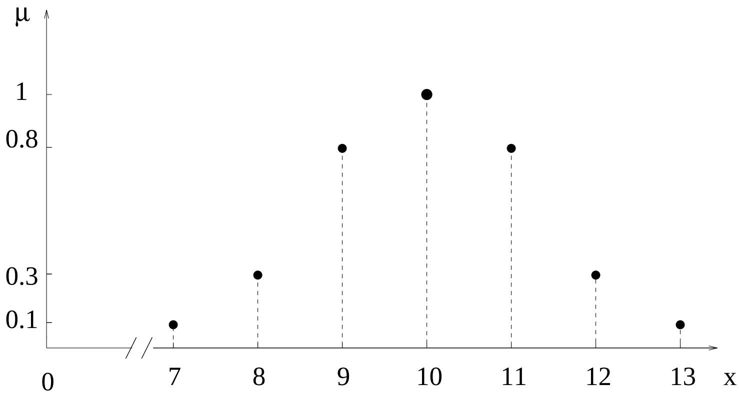

Fig. 1.6. Integers *close* to 10.

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Example 1.8

We have seen in Example 1.5 that the description of *tall men* by classical sets is not adequate. Now we employ for the same purpose the fuzzy set $\mathcal{T} = \{(x, \mu_{\mathcal{T}}(x))\}$, where x measured in cm belongs to the interval [160, 200] and $\mu_{\mathcal{T}}(x)$ is defined by (see Fig 1.7)

$$\mu_{\mathcal{T}}(x) = \begin{cases} \frac{1}{2(30)^2}(x-140)^2 & \text{for} \quad 160 \le x \le 170, \\ -\frac{1}{2(30)^2}(x-200)^2 + 1 & \text{for} \quad 170 \le x \le 200. \end{cases}$$

The membership function $\mu_{\mathcal{T}}(x)$ is a continuous piecewise-quadratic function. The numbers on the horizontal axis x give height in cm and the vertical axis μ shows the degree to which a man can be labeled *tall*. According to the graph in Fig. 1.7, if a person's height is 160 cm, the person is a little tall (degree 0.22), 180 cm stands for almost tall (degree