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Exact solution to free vibration of beams partially supported by an elastic foundation

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Abstract This study pertains to the free vibration problem of beams on an elastic foundation of the Winkler type, which is distributed over a particular length of the beam. Closed form solutions are developed by solving the governing differential equations of beams. Moreover, an innovative mathematical approach is proposed to find the precise analytical solution of the free vibration of beams with mixed boundary conditions. Results are discussed in detail through verification studies. Ultimately, it was concluded that the proposed mathematical method could successfully obtain the exact solution to the free vibration problem of beams on partial elastic foundations under mixed boundary conditions.

1. Introduction

The problem of beams on an elastic foundation by itself has many applications in engineering problems from a general perspective. Many analytical and numerical methods on the beam problem, with various types of foundation, have been conducted. For instance, Chen et al. [1] made use of a mixed method, which combined the state space method and the differential quadrature method for the bending and free vibration of arbitrarily thick beams resting on a Pasternak elastic foundation. Yokoyama [2] presented a finite element technique for determining the vibration characteristics of a uniform Timoshenko beam–column supported on a two-parameter elastic foundation. Cornil et al. [3] expanded nonlinear differential equations for static deflection, and linear differential equations for vibrational motion to analyze the free vibration of a beam that has undergone a large static deflection. Mehri et al. [4] used the Green function to obtain the linear dynamic response of a uniform Euler–Bernouli beam under different boundary conditions excited by a moving load. Amiri and Onyango [5] obtained the simply supported beam response on an elastic foundation carrying repeated rolling concentrated loads by means of a Fourier sine transformation. Particular attention is being paid to the dynamic characteristics of beams over elastic foundations whose modeling is based on the Winkler hypothesis. Many other models have been used to simulate the elastic foundation of both beam and plate problems [6–9], but the Winkler model is often adopted. In this approach, the foundation is modeled using the Winkler model of an elastic foundation in which the vertical displacement is assumed to be proportional to the contact pressure at any point.

The free vibration of beams under various boundary conditions has been extensively investigated for many years. Eisenbegrer et al. [10] treated the problem of the vibration of a beam, with part of it supported by a Winkler type elastic foundation. Williams and Kennedy [11] carried out the vibration analysis of beams, and provided the dynamic member of different stiffness variations for a beam on an elastic foundation with general elastic boundary supports. Kukla [12] studied the free vibration of a beam supported on a stepped elastic foundation under various boundary conditions.

In this paper, an exact solution to the free vibration problem of beams having mixed BCs (i.e. simply-supported, clamped or a combination of both) is proposed. Governing differential equations of beams having underlying elastic springs, which occupy a particular length of the beam, are solved through the use of the Fourier series. The solution is subsequently expanded to the inclusion of clamped BC at one or more edges. A number of validation studies are carried out to verify the accuracy

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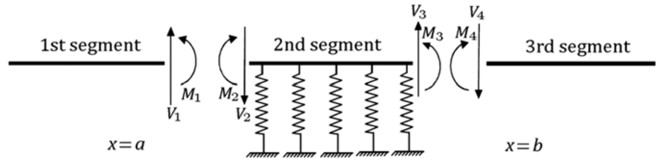

Figure 1: Segmentation of the beam.

of the proposed method. The proposed method has the advantage of directly solving the governing differential equations, which is in contrast to conventional closed form solutions, where continuity equations are added to the existing governing differential equation to find the unknowns of problems having partial elastic foundations. Most importantly, the current method could be readily expanded to plate problems having a partial elastic foundation underneath the plate surface.

2. Exact solution by dividing the beam into separate segments

Figure 1 illustrates a beam partially occupied by uniformly-distributed springs. Analytical approaches have been proposed to solve the governing differential equation of this problem, using the method of THE separation of variables. The conventional methods proposed currently mostly divide this beam into three segments and apply continuity conditions to solve the differential equation. In these methods, the middle segment is usually a beam occupied by an elastic foundation.

It could be possible to write the differential equation of the free vibration of each segment. Doing so for the first segment results in:

$$EI\frac{\partial^4 w_1}{\partial x^4}+\rho A\frac{\partial^2 w_1}{\partial t^2}=0, \tag{1}$$

where E, I, ρ and A are the Young modulus, the moment of inertia of the beam, the mass per unit volume and the area of the beam cross section, respectively.

$w(x,t)$ could be expressed as a product of a function of x and a function of t (i.e. $w(x,t) = F(x)G(t)$). Using differentiation and elimination of the time-dependant terms, the differential equation would be:

$$EI\frac{d^4F_1(x)}{dx^4}-\rho A\omega^2F_1(x)=0, \tag{2}$$

in which, ω is the frequency of vibration.

One possible solution to the differential equation in Eq. (2) is:

$$F_1(x) = C_1\cosh\lambda_1x + C_2\sinh\lambda_1x + C_3\cos\lambda_1x + C_4\sin\lambda_1x, \tag{3}$$

in which:

$$\lambda_1=\sqrt[4]{\frac{\rho A\omega^2}{EI}}, \tag{4}$$

and the coefficients, C_1–$C4$, are the unknowns.

For the middle part, since the beam is on a uniformly-distributed elastic foundation, the differential equation of free vibration takes the form of the following equation:

$$EI\frac{\partial^4 w_2}{\partial x^4}+kw_2+\rho A\frac{\partial^2 w_2}{\partial t^2}=0. \tag{5}$$

Analogously, using the method of separation of variables, it is possible to write:

$$EI\frac{d^4F_2(x)}{dx^4}+kF_2(x)-\rho A\omega^2F_2(x)=0, \tag{6}$$

where k is the elastic foundation stiffness. The solution to this differential equation could be:

$$F_2(x) = C_5\cosh\lambda_2x + C_6\sinh\lambda_2x + C_7\cos\lambda_2x + C_8\sin\lambda_2x, \tag{7}$$

where:

$$\lambda_2=\sqrt[4]{\frac{\rho A\omega^2-k}{EI}}, \tag{8}$$

and the coefficients, C_5–C_8, are the unknowns. Similarly, it is possible to develop a solution for the last part of the beam:

$$F_3(x) = C_9\cosh\lambda_1x + C_{10}\sinh\lambda_1x + C_{11}\cos\lambda_1x + C_{12}\sin\lambda_1x. \tag{9}$$

As can be observed, C_9–C_{12} are the unknowns of the three abovementioned solutions. In order to find these unknowns, it is required to develop twelve equations, which are explicitly obtained using boundary conditions. For simply supported ends, the boundary conditions are as follows:

$$w=0,\quad EI\frac{d^2w}{dx^2}=0. \tag{10}$$

As for clamped ends, we have:

$$w=0,\quad \frac{dw}{dx}=0. \tag{11}$$

And, finally, for free ends:

$$EI\frac{d^2w}{dx^2}=0,\qquad EI\frac{d^3w}{dx^3}=0. \tag{12}$$

There remain eight other equations, which are readily obtained using continuity conditions in the vicinities of the different segment connections. Using Figure 1, these equations for $x = a$ and $x = b$ are as follows:

$$w_1=w_2,\quad \frac{dw_1}{dx}=\frac{dw_2}{dx},$$
$$M_1=M_2\Rightarrow EI\frac{d^2w_1}{dx^2}=EI\frac{d^2w_2}{dx^2},$$
$$V_1=V_2\Rightarrow EI\frac{d^3w_1}{dx^3}=EI\frac{d^3w_2}{dx^3}. \tag{13}$$

Having twelve equations and twelve unknowns, the following system of equations can be developed:

$$[A]_{12\times12}[C]_{12\times1}=[0]_{12\times1}. \tag{14}$$

In this equation, $[A]$ is obtained by using boundary conditions and continuity equations. $[C]$ is the coefficient matrix, which contains the twelve unknowns. In order for this matrix to have a non-trivial solution, the determinant of matrix A is set equal to zero, which leads to the evaluation of the natural frequencies of the vibration of the beam.

3. Proposed solution

The method proposed in this study takes advantage of sinusoidal functions to solve the differential equations, without the beam segmentation explained previously. In this section, the solution to the problem of a simply supported beam on a partial elastic foundation is presented, and then the solution is extended to other types of boundary condition.

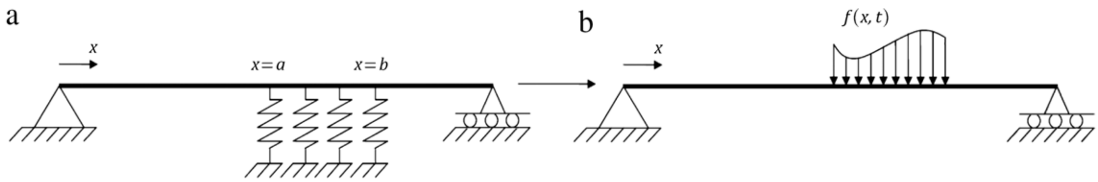

Figure 2: Single span beam supported by partial elastic foundation. (a) Springs as elastic foundation; and (b) modeling of foundation as imposed distributed load.

3.1. Simply-supported beams

Figure 2 shows a simply-supported beam partially occupied by uniformly distributed springs. An alternative approach would be to consider the springs as a distributed load on the beam and solve the problem of a simply-supported beam with a distributed load (Figure 2(b)).

Presuming that the dynamic deflection of the beam is $w(x,t)$, the governing differential equation of this new structure is:

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}=f(x,t). \tag{15}$$

The distributed load is denoted by $f(x,t)$, which could be decomposed into two separate functions through the method of the separation of variables. However, $f(x,t)$ is a function of beam deflection. Thus, it is possible to write:

$$f(x,t)=q(x)G(t)=-kw(x,t), \tag{16}$$

where k is the elastic foundation (i.e. spring) stiffness. Using the method of the separation of variables,

$$w(x,t)=F(x)G(t). \tag{17}$$

Consequently, Eq. (15) can be rewritten as:

$$EI\frac{d^4F(x)}{dx^4}-\rho A\omega^2F(x)=q(x), \tag{18}$$

in which ω is the frequency of vibration. One could express the solution to Eq. (18) by using trigonometric functions as the following:

$$F(x)=\sum_{n=1}^{\infty}a_n\sin\frac{n\pi x}{L}, \tag{19}$$

where L is the length of the beam. The functions given in Eq. (19) suitably satisfy simply-supported BCs. Substituting Eq. (19) into Eq. (18) yields:

$$\sum_{n=1}^{\infty}a_n\left(EI\frac{n^4\pi^4}{L^4}-\rho A\omega^2\right)\sin\frac{n\pi x}{L}=q(x). \tag{20}$$

The Fourier series for $q(x)$ could be defined as:

$$q(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}, \tag{21}$$

in which b_n is the Fourier coefficient, which can be written using Eq. (16), as:

$$b_n=-\frac{2}{L}\int_a^b kF(x)\sin\frac{n\pi x}{L}dx. \tag{22}$$

In the above equation, a and b (shown in Figure 2(a)) are the amounts of variable x in the beginning and end of the elastic foundation occupation beneath the beam. Substituting Eq. (21) into Eq. (20), the relation between coefficients a_n and b_n is obtained as:

$$a_n=\frac{b_n}{\left(EI\frac{n^4\pi^4}{L^4}-\rho A\omega^2\right)}. \tag{23}$$

Finally, after applying Eqs. (22), (19) and (23), a recurrence formula is obtained to relate the two unknown coefficients and provide a sequence for acquiring the frequencies of the beam, as follows:

$$b_n=-\frac{2}{L}\int_a^b k\left[\sum_{m=1}^{\infty}\frac{b_m}{\left(EI\frac{m^4\pi^4}{L^4}-\rho A\omega^2\right)}\sin\frac{m\pi x}{L}\right]\times\sin\frac{n\pi x}{L}dx. \tag{24}$$

In order to present the procedure by which the frequencies are obtained, one should assume that the first n terms of the series in Eq. (24) are expanded. Therefore, there will be 1 equation with n unknowns (i.e. $b_1, b_2, \ldots, b_n$ as unknowns). Naturally, $n-1$ more equations are required for a unique solution to this set of unknowns. If one develops Eq. (24) for other Fourier coefficients (i.e. b_2 up to b_n on the left hand side), after the integration of the coefficients on the right-hand side, it would be possible to establish a system of n equations with n unknowns, as follows:

$$\begin{aligned}b_1&=c_{1,1}b_1+c_{1,2}b_2+\cdots+c_{1,n}b_n,\\ b_2&=c_{2,1}b_1+c_{2,2}b_2+\cdots+c_{2,n}b_n,\\ &\vdots\\ b_n&=c_{n,1}b_1+c_{n,2}b_2+\cdots+c_{n,n}b_n.\end{aligned} \tag{25}$$

In the above equation, $c_{1,1}$–$c_{n,n}$ are coefficients which contain ω. It can be written in a matrix form and factored to generate a set of a homogeneous system of equations:

$$\begin{bmatrix}c_{1,1}-1 & c_{1,2} & \cdots & c_{1,n}\\ c_{2,1} & c_{2,2}-1 & \cdots & c_{2,n}\\ \vdots & \vdots & & \vdots\\ c_{n,1} & c_{n,2} & \cdots & c_{n,n}-1\end{bmatrix}\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix}=\begin{bmatrix}0\\ 0\\ \vdots\\ 0\end{bmatrix}. \tag{26}$$

For a non-trivial solution, the determinant of the coefficient matrix should be equal to zero. The resulting nth-degree, multi-term polynomial would have n roots, which correspond to n frequencies of the vibration of the structure. The accuracy of the solution solely depends upon the number of terms (i.e. n) in Eq. (19), which is taken into account. Obviously, the more the terms are considered, the better the accuracy would be.

3.2. Application to fixed end condition

The procedure described above could be applied readily to clamped end conditions. However, a problem does arise when one wants to use Eq. (19), since it does not satisfy the clamped BC. In order to remedy the problem, an innovative approach is proposed to incorporate the clamped BC into the governing deferential equation in (18) without further need to change the form of the series given in Eq. (19).

Figure 3 illustrates two equivalent structural idealizations in which clamped BC at the top structure was considered to be a combination of a hinge support plus a support reaction, M. This reaction could further be considered as a force couple system

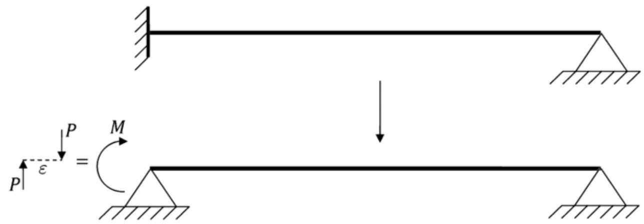

Figure 3: Idealization of clamped BC as simply-supported BC plus a force couple system.

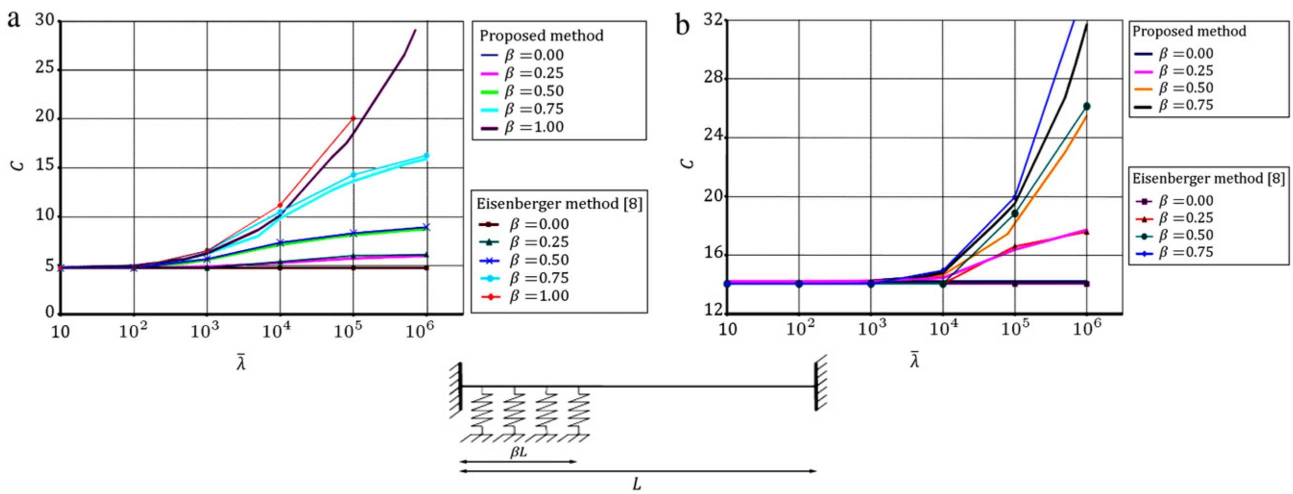

Figure 4: Comparison between results of the present study and those of Eisenbegrer et al. [10]. (a) First mode; and (b) fourth mode.

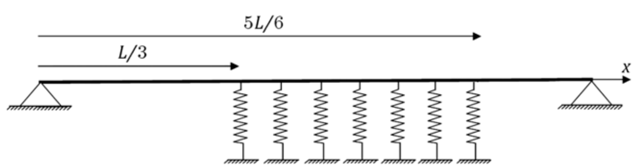

Figure 5: A beam part of which is supported on a distributed elastic foundation.

acting at the end of the beam. The force couple and moment reaction can be related to each other via the equation below:

$$M = \lim_{P\to\infty,\varepsilon\to 0} P \times \varepsilon \tag{27}$$

which simply states that as the coupled forces approach infinity and the distance in between approaches zero, the resulting moment will approach M, which is the fixed end moment induced in the event of clamped BC. Given Eq. (27), it would be possible to proceed with the solution of the governing differential equation in (18). It should be noted that the right-hand side of Eq. (18) is comprised of distributed loads acting on the beam, whereas the force couple system is of concentrated natural force nature. Nevertheless, it can be incorporated in Eq. (18), using the Dirac Delta function as follows:

$$EI\frac{\mathrm{d}^4F(x)}{\mathrm{d}x^4} - \rho A\omega^2 F(x) = q(x)\left(q_C\delta(x=0) - q_C\delta(x=\varepsilon)\right). \tag{28}$$

q_C is an imaginary distributed load caused by clamped BC and:

$$\int_0^L q_C\delta(x=0)\mathrm{d}x = \int_0^L q_C\delta(x=\varepsilon)\mathrm{d}x = P,$$

in which P is shown in Figure 3. It should be noted that the magnitude of q_C and P are equal, but the dimensions are different. In an analogous way to the previous section, the Fourier coefficients for this additional loading should be determined as:

$$c_{n_1} = \frac{2}{L}\int_0^L q_C\delta(x=0)\sin\frac{n\pi x}{L}\mathrm{d}x = 0,$$

$$c_{n_2} = \frac{2}{L}\int_0^L q_C\delta(x=\varepsilon)\sin\frac{n\pi x}{L}\mathrm{d}x = \frac{2}{L}P\sin\frac{n\pi\varepsilon}{L}, \tag{29}$$

where c_{n_1} and c_{n_2} are the Fourier coefficients for $q_C\delta(x=0)$ and $q_C\delta(x=L)$, respectively. Subtracting these two coefficients would result in $-\frac{2}{L}P\sin\frac{n\pi\varepsilon}{L}$, where its limit, when P approaches infinity and ε approaches zero, could be computed using (27), as:

$$\lim_{P\to\infty,\varepsilon\to 0} -\frac{2}{L}P\sin\frac{n\pi\varepsilon}{L} = \lim_{P\to\infty,\varepsilon\to 0} -2P\varepsilon\frac{n\pi}{L^2} = -2M\frac{n\pi}{L^2}. \tag{30}$$

Thus, Eq. (30) can be rewritten using the Fourier series for all the terms as:

$$\sum_{n=1}^{\infty} a_n\left(EI\frac{n^4\pi^4}{L^4} - \rho A\omega^2\right)\sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L} + M\sum_{n=1}^{\infty} -\frac{2n\pi}{L^2}\sin\frac{n\pi x}{L}. \tag{31}$$

Table 1: Verification the first five natural frequencies between present study and exact solution.

$\bar{k}$	$\bar{\omega}$					
	Present study					Exact solution
	$n = 15$	$n = 30$	$n = 50$	$n = 80$	$n = 100$	
			Simply-supported			
	10.25	10.25	10.25	10.25	10.25	10.25
	39.54	39.54	39.54	39.54	39.54	39.54
10	88.85	88.85	88.85	88.85	88.85	88.85
	157.92	157.92	157.92	157.92	157.92	157.92
	246.75	246.75	246.75	246.75	246.75	246.75
	13.21	13.21	13.21	13.21	13.21	13.21
	40.11	40.11	40.11	40.11	40.11	40.11
10^2	89.11	89.11	89.11	89.11	89.11	89.11
	158.07	158.07	158.07	158.07	158.07	158.07
	246.83	246.83	246.83	246.83	246.83	246.83
	28.63	28.63	28.63	28.63	28.63	28.63
	45.73	45.73	45.73	45.73	45.73	45.73
10^3	91.67	91.67	91.67	91.67	91.68	91.68
	159.49	159.49	159.49	159.49	159.49	159.49
	247.64	247.64	247.64	247.64	247.64	247.642
			Clamped-hinged			
	16.13	15.90	15.81	15.70	15.70	15.70
	51.49	50.71	50.36	50.28	50.23	50.02
10	107.47	105.73	105.20	104.78	104.70	104.27
	184.00	180.85	179.72	179.21	179.02	178.28
	281.17	276.06	274.33	273.39	273.17	272.03
	18.41	18.19	18.18	18.12	18.08	18.02
	52.00	51.22	50.95	50.71	50.73	50.53
10^2	107.65	105.90	105.30	104.95	104.89	104.46
	184.04	181.10	179.90	179.34	179.15	178.41
	281.24	276.14	274.42	273.47	273.26	272.12
	33.50	33.33	33.28	33.24	33.22	33.17
	56.75	56.04	55.71	55.62	55.54	55.33
10^3	109.52	107.88	107.20	106.91	106.81	106.39
	185.41	182.22	181.15	180.59	180.40	179.66
	282.08	276.92	275.20	274.33	274.05	272.91
			Fully clamped			
	23.81	23.20	22.93	22.79	22.74	22.56
	65.61	63.45	62.75	62.35	62.22	61.72
10	128.33	124.53	123.00	122.19	121.93	120.92
	213.76	205.74	203.27	201.96	201.53	199.87
	318.57	307.87	303.83	301.74	301.08	298.56
	25.38	24.80	24.55	24.42	24.37	24.20
	66.04	63.89	63.18	62.79	62.66	62.16
10^2	128.52	124.73	123.20	122.40	122.14	121.12
	213.88	205.86	203.40	202.08	201.66	200.00
	318.64	307.94	303.89	301.82	301.16	298.63
	37.49	37.02	36.83	36.72	36.68	36.55
	70.20	68.11	67.43	67.05	66.93	66.44
10^3	130.51	126.74	125.22	124.42	124.16	123.16
	215.09	207.85	204.62	203.30	202.88	201.22
	319.34	308.64	304.59	302.51	301.85	299.32

Consequently, it leads to the following:

$$a_n = \frac{b_n}{EI\frac{n^4\pi^4}{L^4} - \rho A\omega^2} + \frac{M\left(-\frac{2n\pi}{L^2}\right)}{\frac{EIn^4\pi^4}{L^4} - \rho A\omega^2}. \qquad (32)$$

It is of interest to investigate the difference between Eqs. (23) and (32), which is the presence of an additional term containing M on the right-hand side. It does incorporate another unknown into the problem, for which one additional equation is required to reach a unique solution. To reach this additional equation, one could differentiate the series in Eq. (19) to satisfy the BC part of the problem. At the same time, it is well understood that the deflection slope of the beam should equal zero at the clamped BC, as found in the following:

$$\left.\frac{\mathrm{d}F(x)}{\mathrm{d}x}\right|_{x=0} = \sum_{n=1}^{\infty} a_n \frac{n\pi}{L} \cos \left.\frac{n\pi x}{L}\right|_{x=0} = 0. \qquad (33)$$

From Eq. (33), it is obvious that $\sum_{n=1}^{\infty} a_n \frac{n\pi}{L} = 0$, thereby:

$$\sum_{n=1}^{\infty}\left(\frac{b_n}{EI\frac{n^4\pi^4}{L^4} - \rho A\omega^2} + \frac{\frac{-2Mn\pi}{L^2}}{EI\frac{n^4\pi^4}{L^4} - \rho A\omega^2}\right)\frac{n\pi}{L} = 0. \qquad (34)$$

Eqs. (32) and (34) constitute $(n+1)$ equations required to solve the $(n+1)$ unknowns existing in the set of recurrence formulas generated by Eq. (32). The same procedure could be applied to another clamped BC, with the additional force couple system on the right support (Figure 3), where it naturally invokes another unknown (e.g. M_2) to the set of equations. Nonetheless, using a similar technique in Eq. (33), for $x = L$, the required equation is acquired, and then it would be possible to solve the resulting $(n + 2)$ simultaneous equations.

4. Verification studies

In this section, initially, the analytical method proposed in this study is validated against the work conducted by Eisenbegrer et al. [10]. Natural frequencies of vibrations and mode shapes for a single span beam with various BCs can be calculated using the procedure described earlier. However, to summarize the results of the comparisons, a general case is selected, which represents the inclusion of two fixed BCs in the proposed formulation (i.e. fully fixed beam). Figure 4(a) and (b) indicate the results of the first and fourth modes of vibration compared with the work conducted by Eisenbegrer et al. [10]. These results are presented in the same non-dimensional form as in [10], with logarithmic abscissa, that is:

$$\bar{\lambda} = \frac{kL^4}{EI}, \qquad c = L\sqrt[4]{\frac{\omega^2 \rho A}{EI}}. \tag{35}$$

Using fifteen terms of Eq. (19), it is seen that there exists good agreement between the results of the proposed method and those of Eisenbegrer et al. [10].

Using the exact method given in Section 2, it is also possible to validate the proposed solution to the free vibration problem of beams on a partial elastic foundation having different boundary conditions. Three different boundary conditions, namely, simply-supported, clamped-hinged and fully clamped beam problems, are considered on a partial elastic foundation distributed from $\frac{L}{3}$ up to $\frac{5L}{6}$ of the beam length in which L is the beam length (Figure 5). The dimensionless stiffness parameter is taken to have three different values. The first five natural frequencies of the beam were obtained, using the two methods, and given in Table 1. In this table:

$$\bar{\omega} = \omega\sqrt{\frac{\rho A L^4}{EI}}, \qquad \bar{k} = \frac{kL^4}{EI}, \tag{36}$$

in which ω is the frequency of vibration and k is the elastic foundation stiffness.

In order to evaluate the effect of the number of terms on the resulting frequencies, five different cases were considered in each of which the number of terms in Eq. (19) is different. n is the number of terms in Eq. (19).

Close agreement between the two sets of results demonstrates the validity of the proposed analytical approach for beams. As can be seen in the results, better accuracy can be reached once the number of terms increases.

5. Conclusion

A novel analytical solution to the free vibration problem of beams on partial elastic foundations was presented in this study. A simple and robust method for incorporating clamped BC into the governing differential equation of vibration of such structures was also provided and verified against available research found in literature and against the exact solution. Basically, it was shown that by superposition of appropriate trigonometric functions, not only can underlying springs be incorporated as imposing loads into governing differential equations, but also clamped BC, as an external applied force couple, can be imparted to the solution of differential equations.

Thus, based on the overall parametric studies provided in this study, it could be stated that the proposed method successfully calculates the natural frequencies of beams, where only part of them is supported by an elastic foundation.

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